Eta invariant : Lecture 18

Last time I discussed the principal-bundle approach to the families index. Today I want to continue the discussion of the odd families index. Recall that I stated (but did not really prove) that

$$G^{-\infty}(X;E) = \{A \in \Psi^{-\infty}(X;E);\ (Id+A)^{-1} = Id + B,\ B \in \Psi^{-\infty}(X;E)\}$$

is a classifying space for odd K-theory. For the moment I want to concentrate on $\pi_1(G^{-\infty})$, or really $H^1(G^{-\infty})$. The latter is generated by the Fredholm determinant

$$\det : Id + \Psi^{-\infty}(X,E) \longrightarrow \mathbb{C}$$

$$G^{-\infty}(X;E) = \{A \in \Psi^{-\infty}(X;E);\ \det(Id+A) \neq 0\}$$

(1)
$$c_1 = \frac{1}{2\pi i}\, d \log \det$$
$$= \frac{1}{2\pi i} \operatorname{Tr}\left((Id+A)^{-1} dA\right).$$

Then $[c_1]$ spans $H^1(G^{-\infty})$.

The corresponding loop group

$$\mathcal{L}G^{-\infty} = \{F : S^1 \to G^{-\infty};\ F(1) = \mathbb{1}\}$$

is a classifying group for even K-theory. Let me "flatten" $\mathcal{L}G^{-\infty}$ a little by looking at

$$\Psi_s^{-\infty}(X;E) = \{\mathcal{S}(\mathbb{R};\Psi^{-\infty}(X;E))$$
$$\equiv \mathcal{S}(\mathbb{R}\times X^2;\ \mathrm{Hom}(E)\otimes\Omega_n)\}$$

and the corresponding group

$$G_s^{-\infty}(X;E) = \{A \in \Psi_s^{-\infty}(X;E);\ (\mathbb{1}+A)^{-1} = \mathbb{1}+B,\ B \in \mathcal{S}(\mathbb{R};\Psi^{-\infty})\}.$$

<u>Lemma</u> $G_s^{-\infty}(X;E)$ is also a classifying group for even K-theory.

<u>Proof</u> Taking the 1-point compactification $\mathbb{R} \to S$, $\infty \mapsto 1$ allows us to identify

$$\Psi_s^{-\infty}(X;E) \cong \{A \in C^\infty(S;\Psi^{-\infty}(X;E));\ A = 0 \text{ at } 1\}$$

Then it is relatively easy to see that $G_s^{-\infty}(X;E) \to \mathcal{L}G^{-\infty}$

as a deformation retract.

<u>Exercise</u> Write this out carefully!

Now suppose we have a family of self-adjoint elliptic pseudo-differential operators $P_y \in C^\infty(Y; \Psi^1(X;E))$ $P_y^* = P_y$ for some vector bundle E and volume form.

<u>Lemma</u> If $P \in \Psi^1(X;E)$ is elliptic and self-adjoint there exists $a \in \Psi_s^{-\infty}(X;E)$ s.t.

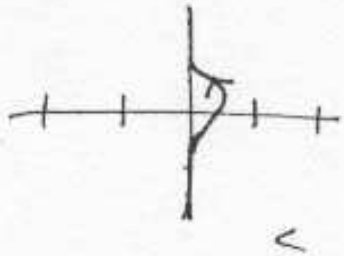
$$P + i\tau + a(\tau) \in \Psi^1(X;E) \text{ is invertible } \forall \tau \in \mathbb{R}.$$

<u>Proof</u> Since P is elliptic and self-adjoint it has discrete spectrum $sp(P) \subset \mathbb{R}$. It follows that $P + i\tau$ is invertible for all $\tau \in \mathbb{R} \setminus \{0\}$ and the only problem is the finite dimensional null space. Let $\Pi_0 \in \Psi^{-\infty}(X;E)$ be projection onto this null space then we choose $\varphi \in C_c^\infty(\mathbb{R})$ s.t.

$$(P + i\tau + \varphi(\tau)\Pi_0)^{-1} \in \Psi^{-1}(X;E) \ \forall \tau \in \mathbb{R}.$$

where φ is a curve like this

$i\tau + \varphi(\tau)$:



Exercise Find an explicit φ.

Now we can consider the principal bundle

$$\mathcal{P}_y = \{A \in \Psi_s^{-\infty}(X;E);\ P_y + i\tau + A(\tau) \text{ is invertible in } \Psi^{-1}(X;E)\ \forall \tau \in \mathbb{R}\}$$

This is a non-trivial principal bundle with structure group $G_s^{-\infty}(X;E)$.

$$(P_y + i\tau + A_1(\tau))^{-1}(P_y + i\tau + A_2(\tau)) = I + B(\tau),\quad B \in \Psi_s^{-\infty}(X;E)$$

if $A_1, A_2 \in \mathcal{P}_y$. Since the ~~classifying~~ structure group is a classifying group for even K-theory we "know" that the obstruction to the triviality of

$$\begin{array}{ccc} G_0^{-\infty}(X;E) & - & P \\ & & \downarrow \\ & & Y \end{array}$$

PB'

is a class $\mathrm{Ind}_0(P) \in K^{-1}(Y)$. To construct the class directly we need a short exact sequence

(M) $\quad G_s^{-\infty}(X;E) \longrightarrow G \longrightarrow H_0$

where H_0 denotes the 1st K-theory of G as (weakly) contractible. This always exists, but I will do it explicitly later.

For the moment think about the 1st odd Chern class. We have also seen that $c_1 \in H^1(H_0)$ exists, corresponding to the Fredholm determinant. Thus if $\mathrm{Ind}_0(P) \in [Y; H_0]$ can be constructed corresponding to the non-triviality of (PB) we should be able to "find" $\mathrm{Ind}_0^*(P)^* c_1 \in H^1(Y)$ from duality. That is, we should be able to construct

a 'determinant function' $\tau : Y \to \mathbb{C}^*$ s.t.

$$\mathrm{Td}_0(P)^{\vee} c_1 = \left[\frac{1}{2\pi i} d \log \tau \right].$$

Where will this come from? Look at the (unknown to us at the moment) sequence (II). On the left we have classes in all even dimensions, on the right in all odd dimensions and in the middle, nothing. The bottom class, measuring the components, is the winding number (at right)

$$w(F) = \frac{1}{2\pi i} \int_S \frac{d}{d\theta} \log \det (F(\theta))\, d\theta$$

$F : S^1 \to G^{-\infty}, \mathrm{Id}$ or $F = \mathrm{Id} + A(t)$, $A \in \Psi_s^{-\infty}(X; E)$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d}{dt} \log \det (\mathrm{Id} + A(t))\, dt$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}\left((\mathrm{Id} + A(t))^{-1} \frac{dA}{dt} \right) dt.$$

<u>Lemma</u> The winding number is a group homomorphism

$$w : G_s^{-\infty}(X;E) \twoheadrightarrow \mathbb{Z}.$$

<u>Proof</u> Thus as the multiplication of the determinant:

$$\log \det (F(z)G(z)) = \log \det F(z) + \log \det G(z)$$

$$\Rightarrow \quad w(FG) = w(F) + w(G)$$

That w takes values in $\mathbb{Z}$ follows from the properties of the logarithm; the only surjective requirement comes to find a case with $w(F)=1$; e.g. a rotation in a variable.

So, now to the construction. We want to find

$$\mathfrak{g} : \mathcal{P} \longrightarrow \mathbb{C} \quad \text{s.t.}$$

$$\mathfrak{g}(FP_s) = \mathfrak{g}(P_s) + w(F),$$
$$\forall F \in G_s^{-\infty}(X;E).$$

I will use the following properties of

$$P : \mathbb{R}_\varepsilon \to \Psi^h(X;E) \quad C^\infty$$

i) $P^{-1} : \mathbb{R}_\varepsilon \to \Psi^{-h}(X;E)$ exists,

ii) $\frac{d^\ell}{d\varepsilon^\ell} P : \mathbb{R} \to \Psi^{h-\ell}$,

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For any continuous seminorm $\| \ \|_{(k)}$ on $\Psi^{\ell}(X;E)$

(SE) $$\left\| (1+|\tau|)^{j} \frac{d^{\ell}}{d\tau^{\ell}} P(\tau) \right\|_{(k-\ell+r)} \text{ is bdd}$$

together with the existence of the provided asymptotic expansions.

Then we just try to regularize the formula for $\omega(F)$:

$$\omega(F) = \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}\left(F^{-1}(\tau) \frac{dF(\tau)}{d\tau} \right) d\tau .$$

The basic problem is that

$$\mathrm{Tr} : \Psi^{-n-1}(X;E) \to \mathbb{C}$$

does not extend to Ψ^{n} to vanish on commutators.

Instead we regularize in τ, using (SE).

The estimates (SE) mean that ~~for N sufficiently large,~~

$$\left(\frac{d}{d\tau}\right)^{N} \left(F^{-1}(\tau) \frac{dF(\tau)}{d\tau} \right) \in C^{\infty}(\mathbb{R}; \Psi^{-N-1}(X;E))$$

Thus, for $N > \dim X$ we can take the trace and consider

$$\varphi_N(\tau) = \mathrm{Tr}\left(\left(\frac{d}{d\tau}\right)^N F^{-1}(\tau)\frac{dF}{d\tau}(\tau)\right) \qquad N > \dim X$$

Thus $\frac{d}{d\tau}\varphi_N(\tau) = \varphi_{N+1}(\tau)$. Now, if we integrate from the origin N times

$$\psi_N(\tau) = \int_0^{\tau_N} \cdots \int_0^{\tau_1} \varphi_N(\tau)\, d\tau$$

we get a smooth function which still depends on N but clearly

$$\psi_{N'}(\tau) = \psi_N(\tau) + p_{N'N}(\tau),$$

$p_{N'N}(\tau)$ a polynomial.

<u>Claim</u> For $F(\tau) = A + i\tau + a(\tau) \in \mathcal{P}$, $A \in \Psi^1(X;E)$ elliptic and self-adjoint,

$$\psi_N(\tau) \sim \sum_{j=0}^{\infty} \tau^{L-j} a_j^{\pm} + \log\tau \cdot p_N^{\pm}(\tau)$$

as $\tau \to \pm\infty$ has a <u>complete asymptotic</u> expansion.

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<u>Definition</u> Subject to the validity of this claim we define

$$\xi(F(\cdot)) = \frac{1}{2\pi i} \times \text{coeff of } \hbar^0 \text{ in expansion of}$$

$$\int_{-T}^{T} \psi_N(\tau_N)\, d\tau_N \text{ as } T \to$$

<u>NB</u>. This does <u>not</u> depend on N, since replac ψ_N by $\psi_{N'}$ changes the integral by a polynomial

$$\int_{-T}^{T} p(t)\, dt = T\,Q(T)$$

has <u>no</u> constant term as $T \to \infty$. This comes f the 'choice' of symmetry $[-T, T]$ of the interval of integration.

<u>Lemma</u>: ξ has the properties

$$\xi(A(\tau)F(\tau)) = \omega(A(\tau)) + \xi(F(\tau))$$

if $A \in G_s^{-\infty}(X; E)$.

<u>Proof</u>. Expand out

$$\left(\frac{d}{d\tau}\right)^N \left(F^{-1}(\tau)\, A^{-1}(\tau)\, \frac{d}{d\tau}\left(A(\tau)\, F(\tau)\right)\right)$$

$$= \left(\frac{d}{d\tau}\right)^N \left(F^{-1}(\tau)\, \frac{dF}{d\tau}\right) + \left(\frac{d}{d\tau}\right)^N \left(F^{-1}(\tau)\, A^{-1}(\tau)\, \frac{dA}{d\tau}\, F(\tau)\right)$$

observe that the second term is divisible in $\Psi_0^{-\infty}(X;E)$. Thus

$$\xi(A(\tau)F(\tau)) - \xi(F(\tau))$$

$$= \frac{1}{2\pi i} \text{ coeff of } T^0 \text{ in } \int_{-T}^{T}\int_{0}^{\tau_N}\cdots\int_{0}^{\tau_1} \left(\frac{d}{d\tau}\right)^N \Big(\underbrace{\mathrm{Tr}\left(F^{-1}A^{-1}\frac{dA}{d\tau}F\right)}_{\mathrm{Tr}\left(A^{-1}\frac{dA}{d\tau}\right)}\Big)\, d\tau$$

$$= \frac{N!}{2\pi i} \times \text{coeff of } T^0 \text{ in } \int_{-T}^{T} \mathrm{Tr}\left(A^{-1}\frac{dA}{d\tau}\right) d\tau$$

$$= w(A).$$

since the commutation is good enough for operators of order $-\infty$.

<u>Exercise</u> Show that if F and G are two 'admissible' elliptic families then $\xi(FG) = \xi(F) + \xi(G)$.

<u>Claim</u> $d\xi = \pi^* \chi$ for $\chi \in C^\infty(Y; \Lambda^1)$, $d\chi = 0$, and $[\chi] = [c_1] \equiv \mathrm{Ind}_0(P)^* c_1$ in $H^1(Y)$.