Eta invariant: Lecture 18

Last time I discussed the principal-bundle approach to the families index. Today I want to continue the discussion of the old families index. Recall that I stated (but did not really prove) that

\[ G^{-\infty}(X; E) = \left\{ A \in \Psi^{-\infty}(X; E) \mid (I + A)^{-1} = I + B, B \in \Psi^{-\infty}(X; E) \right\} \]

is a classifying space for old $K$-theory. For the moment I want to concentrate on $\tilde{H}^1(G^{-\infty})$, or really $H^1(G^{-\infty})$. The latter is generated by the Fredholm determinant

\[ \det: I + \Psi^{-\infty}(X; E) \to \mathbb{C} \]

\[ G^{-\infty}(X; E) = \left\{ A \in \Psi^{-\infty}(X; E) \mid \det((I + A)^{-1}) \neq 0 \right\} \]

(1)

\[ C_1 = \frac{1}{2\pi i} \int \log \det \]

\[ = \frac{-1}{2\pi i} \text{Tr} \left( (I + A)^{-1} dA \right). \]

Thus $[C_1]$ spans $H^1(G^{-\infty})$. 

\[ \frac{1}{2\pi i} \]
The corresponding loop group

\[ L_k^\infty = \{ F : \mathbb{S}^1 \to G^\infty ; F(1) = 1 \} \]

is a classifying group for each $K$-theory. Let me "demon" $L_k^\infty$ a little by looking at

\[ \Psi_s^\infty (X; E) = \{ \delta \in \mathcal{A}(R ; \Psi^\infty (X; E)) \}
\]

\[ = \delta (R \times \mathbb{R}^2 ; \text{Hom}(E) \otimes \mathbb{R}^2) \]

and the corresponding group

\[ G_s^\infty (X; E) = \{ A \in \Psi_s^\infty (X; E) ; (I \pm A)^{-1} \in \text{Id} \pm R, B \in \mathcal{A}(R ; \Psi^\infty) \} \]

Lemma $G_s^\infty (X; E)$ is also a classifying group for each $K$-theory.

Proof: Taking the 1-point compatification \( R \rightarrow \mathbb{S} \),

\[ x \rightarrow 1 \] allows us to identify

\[ \Psi_s^\infty (X; E) \cong \{ A \in C^\infty (\mathbb{S} ; \Psi^\infty (X; E)) ; A = 0 \text{ or } 1 \} \]

Then it is relatively easy to see \( \Psi_s^\infty (X; E) \rightarrow L_k^\infty \).
Exercises Write these at a bit more carefully!

Now suppose we have a family of surf object

\[ \Phi : \mathcal{C}^0(Y; \Omega^1(X; E)) \]

\[ \Phi^* \text{ for some inner product on } E \text{ an inner form} \]

Lemma If \( \Phi \in \mathcal{C}^1(X; E) \) is smooth and surf object

Then exists \( a \in \mathcal{C}^{0,\infty}(X; E) \) s.t.

\[ P + i a \in \mathcal{C}^* \]

\[ \forall \tau \in \mathbb{R} \]

Proof Since \( P \) is smooth and surf object as an
domain spectrum \( \text{sp}(P) \subset \mathbb{R} \).

It follows that \( P + i \tau \) is smooth for \( \tau \in \mathbb{R} \) for all \( \tau \in \mathbb{R} \).

As only satisfy in the finite dimensional with space.

Let \( \pi \in \mathcal{C}^{0,\infty}(X; E) \) be a function of the

null space then we claim \( \phi \in \mathcal{C}_{0,\infty}(\mathbb{R}) \) s.t.

\[ (P + i \tau + \phi(\tau) \pi) \in \mathcal{C}^1(X; E) \forall \tau \in \mathbb{R} \]
Then as a non-trivial principal bundle with structure group $G_\infty (X; E)$,

$$ (P_y + i\tau + A_y(\tau))^\dagger (P_y + i\tau + A_y(\tau)) $$

$$ = 1 + B(\tau), \quad B \in \mathfrak{g}_\infty (X; E) $$

Since the structure group is a classifying group for even K-theory we "know" that the obstruction to the triviality of
$G_{\infty}(\mathcal{E}) - P$

is a ten $\mathcal{F} \text{Ind}_0(P) \in K^{-1}(Y)$. To construct
the direct image we need a short exact sequence

$\mathcal{H}^1$

$G_{\infty}(\mathcal{E}) \rightarrow G \rightarrow H_0$

where $H_0$ denotes the $K$-theory of $G$ is (weakly)
contractible. The always exists, but I will return
exactly later.

For the moment think about the 1st odd
Chern class. We have already seen that $c_1 \in H^1(H_0)$
exists, corresponding to the Fredholm determinant.

Then if $\text{Ind}_0(P) \in \mathcal{G}$ can be constructed
analogously to the non-abelian of (PB) can be shown
to be a “fixed” $\text{Ind}_0(P)^* c_1 \in H^1(Y)$ from
directly. That is, we should be able to construct
a 'determinant function' \( r : Y \to C^* \) s.t.

\[
\text{Ind}_0(P) \cdot c_1 = \left[ \frac{1}{2\pi i} \int \log r \right].
\]

Where will this come from? Look at the (unknown to us at the moment) sequence (II). On the left we have classes in all even dimensions, on the right we have classes in all odd dimensions and on the middle, nothing. The bottom class, measuring the component to the winding number (or index)

\[
W(F) = \frac{1}{2\pi i} \int \frac{d}{d\theta} \log \det (\mathcal{T}_B F(\theta)) \, d\theta
\]

\( F : S^1 \to G \), Id \& \( F = J \tau + \alpha(t), \alpha \in C^\infty, (x, t) \)

\[
= \frac{1}{2\pi i} \int \frac{d}{dt} \log \det (I + \alpha(t)) \, dt
\]

\[
= \frac{1}{2\pi i} \int \text{Tr} \left( \left( I + \alpha(t) \right)^{-1} \frac{d}{dt} \alpha(t) \right) \, dt.
\]
Lemma  The moduli number is a group homomorphism:

$$w: G_\infty^0 (X,E) \rightarrow \mathbb{Z}.$$ 

Proof. This is the multiplicative of the determinant:

$$\log \det (F(z)G(z)^{-1}) = \log \det F(z) + \log \det G(z)$$

$$\Rightarrow \quad w(FG) = w(F) + w(G).$$

That $w$ takes values in $\mathbb{Z}$ follows from the fact that the logarithm of the determinant is a linear map. Any set $\{\Phi(z)\}$ of such maps can be used to construct a curve with $w(F_1) = 1$, e.g. a punctured curve.

So, now to the construction. We want to fit

$$\Phi: B \rightarrow C \quad \text{s.t.}$$

$$\Phi(F_{P_0}) = \Phi(P_0) + w(F),$$

for $F \in G_\infty^0 (X,E)$.

I will use the following facts for it:

$$P: \mathbb{R} \rightarrow \Psi^h (X,E) \quad \text{for}$$

i) $P^{-1}: \mathbb{R} \rightarrow \Psi^h (X,E)$ exists,

ii) $\frac{d}{dt} P: \mathbb{R} \rightarrow \Psi^{h-1}$.
For any continuous semi-norm \( a \| \cdot \|_{\alpha} \), consider the estimate (SE)  
\[
(\text{SE}) \quad \| (1+|t|) \int \frac{d}{dt} P(t) \| \leq C (k \Phi E + \Gamma)
\]

provided together with the existence of certain expansions.

Now we just try to regularize the formula for \( \omega(F) \):

\[
\omega(F) = \frac{1}{2\pi i} \int_{\mathcal{R}} \text{Tr} \left[ F^{-1}(t) \frac{d}{dt} F(t) \right] dt
\]

The basic formula is 161

\[
\text{Tr}: \Psi^{r-n-1}(X, E) \to \mathbb{F}
\]

does not extend to \( \Psi^n \) to want a commutator.

Indeed, we regularize it \( \cong \), using (SE).

Instead, we regularize \( \cong \), using (SE).

The estimate (SE) means that for \( N \) sufficiently large,

\[
\left( \frac{d}{dt} \right)^N \left( F^{-1}(t) \frac{d}{dt} F(t) \right) \in C^0(\mathbb{R}, \Psi^{-N-1}(X, E))
\]
Thus, for $N > d X$ we can take it true at once.

$$\varphi_N(r) = \left( \frac{d}{dr} \right)^N \left( 1 (r) \right)^N \Rightarrow \left( \frac{d}{dr} \right)^N \left( 1 (r) \right)^N = \varphi_{N+1}(r) \quad N > d X$$

Thus $\frac{1}{d X} \varphi_N(r) = \varphi_{N+1}(r)$. Now, if we integrate from the origin $N$ times,

$$\varphi_N(r) = \int_0^r \int_0^r \ldots \int_0^r \varphi_N(r) dr$$

we get a smooth function which still depends on $N$ but clearly

$$\varphi_N(r) = \varphi_N(r) + \varphi_N(r)$$

Claim: For $F(r) = A + i r^2 + a(r) \in \mathbb{C}$, $A \in \mathbb{C}$, self-adjoint,

$$\varphi_N(r) \sum_{j=0}^{N-1} r^{-j} a_j r^j + \log r \cdot \frac{\varphi_N(r)}{r}$$

as $r \to 0$ is a complete asymptotic expansion.
Definition Subject to the validity of the claim we define

\[ \xi(F(t)) = \frac{1}{2\pi i} \times \text{coeff of } F^0 \text{ in expansion of } \]

\[ \int_{-T}^{T} \psi_N(\tau) \int_{-2\pi}^{2\pi} a_{\tau} \, d\tau \, d\tau \]

\[ \int_{-T}^{T} \phi(t) \, dt = T \xi(t) \]

NB. This does not depend on \( N \), since reflecting \( \psi_N \) by \( \psi_N^* \) changes the integral by a polynomial:

\[ \int_{-T}^{T} \phi(t) \, dt = T \xi(t) \]

has no constant term as \( T \to \infty \). The cases of the 'chain' of symmetries \([-T, T]\) of the interval of integration.

Lemma: \( \xi \) satisfies the property

\[ \xi(A(t)F(t)) = \omega(A(t)) + \xi(F(t)) \]

of \( A \in G^0_s \) \( (x, F) \).

Proof. Explain us
\[
\left( \frac{\partial}{\partial z} \right)^N \left( F^{-1}(z) A^{-1}(z) \frac{1}{2} (A(z) F(z)) \right)
= \left( \frac{1}{2} \right)^N \left( F^{-1}(z) \frac{dF}{dz} \right) + \left( \frac{1}{2} \right)^N \left( F^{-1}(z) A^{-1}(z) \frac{dA}{dz} F(z) \right)
\]

Observe that the second term is density \( \Psi_n^*(x, E) \). Thus
\[
\mathcal{S}(A^{-1}(z) F(z)) - \mathcal{S}(F(z))
\]
\[
= \frac{1}{2\pi i} \text{cont} \left( \mathcal{S}^0 \iint \cdots \iint \left( \frac{1}{2} \right)^N \left( \text{Tr} \left( F^{-1} \frac{dF}{dz} \right) \right) \right)
\]
\[
\mathcal{S} \left( A^{-1} \frac{dA}{dz} \right)
\]
\[
= \mathcal{S} \cdot \mathcal{S} \quad \text{and} 
\]
\[
= w(A).
\]

Since the commutator is justified for operators of the form \( A = e^{x + \text{cont} \mathcal{T}^0} \int \cdots \int \text{Tr} \left( A^{-1} \frac{dA}{dz} \right) dz \),

Exercise: Show that \( \Psi(AG) = \Psi(A) \Psi(G) \) and \( \Psi(\text{Adm}) \Psi(\text{Adm}) \Psi(\text{Adm}) \).

Claim: \( \mathcal{S} = \mathcal{S} \) for \( x \in C^0(Y, X) \), \( \mathcal{S} = 0, \) and \( \mathcal{S} \mathcal{L} X = \mathcal{S} \mathcal{L} Y \).