Conormal distributions

The aim today is to properly introduce the space $\Psi_b^k(X)$ for $k \in \mathbb{R}$ and describe as many of the properties as I can get to. A little bit of standard geometry to start with.

A closed subset $Y \subset X$ is an embedded submanifold in a compact manifold (for the moment without boundary) if $C(X)|_Y = C^\infty(Y)$ is a C^∞ structure on Y. This is equivalent to the existence at each point of Y of local coordinates

$$(E) \qquad \begin{array}{l} y_1, \dots, y_p, z_1, \dots, z_{n-p} \text{ on } U \subset X \\ \text{s.t. } Y \cap U = \{z_i = 0\}. \end{array}$$

<u>Theorem</u> [Collar neighbourhood] For any embedded submanifold in a compact manifold without boundary there is an open neighbourhood $Y \subset U \subset X$ and $O_Y \subset U' \subset NY$, of the zero section of the normal bundle, and a diffeomorphism

$$F : U' \longrightarrow U$$

such that

1. $F(y) = y \ \forall y \in Y$, so $F : Y \to O_Y$ is the natural map

2. $F_* : NY \longrightarrow NO_Y$ is the natural map

and any two such maps are homotopic through a smooth family with these properties.

Note that $N_y Y = T_y X / T_y Y$ and for $y \in O_Y \cong Y$ $T_y(N_y Y) = N_y Y \cong N_y O_Y$ since the fibre is complementary to the zero section.

<u>Proof</u> I am not going to do this in detail since it is very standard. The cleanest proof I know of uses a Riemann metric on X and fixes F through the exponential map. The Riemann metric allows $N_y Y$ to be identified with the orthocomplement $(T_y Y)^\perp \subset T_y X$ and then one can check readily that

$$F(y, v) = \exp_y(v)$$

has the desired properties. Thus $F(y,v)$ is the point at parametric distance one along the geodesic with initial point y and initial tangent vector v, or if you prefer at distance $|v|$ for the geodesic with initial vector $v/|v|$.

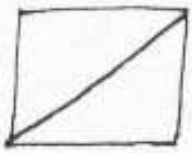
The 'uniqueness' part of the end can be based on the existence of such a 'normal fibration' for the diagonal in X.

In the case of a submanifold $Y \subset X$ of a compact manifold with corners we assume that Y is a p-submanifold (the p- is for product). This is the condition that near each $y \in Y$ there are local coordinates, of the usual adapted sort, $x_1, \dots, x_k, y_1, \dots, y_{n-k}$ in terms of which Y is locally

(P) $\qquad x_1 = \dots = x_j = 0, \quad y_1 = \dots = y_{p-j} = 0.$

Usually we demand $j = 0$, in which case Y is an <u>interior</u> p-submanifold (otherwise it is an interior p-submanifold of some boundary face).

<u>Exercise</u> Note that if we don't assume (P) explicitly but just require that $C^\infty(X)|_Y = C^\infty(Y)$ be the C^∞ structure we allow things like $\{x = x'\}$ in $[0,1]^2$:



Show how to recover (P) locally by assuming just that $C^\infty(X)|_Y = C^\infty(Y)$ and some condition of I_x for

the embedding $I: Y \hookrightarrow X$.

<u>Theorem</u> The collar neighbourhood theorem goes through unchanged for a p-submanifold of manifold with corners.
(closed) compact

Forget the boundary case for the moment, this is not the important part.

We introduce spaces of conormal distributions associated to an embedded submanifold $Y \subset X$. These are distributions on X, singular only at Y. Put

$$\mathcal{V}(X;Y) = \{V \in C^\infty \text{ vector fields on } X, \text{ tangent to } Y\}$$

<u>Exercise</u> Show that in local coordinates $(\mathbb{B})$, $\mathcal{V}(X;Y)$ is spanned by $\partial/\partial y_j$, $y_i \partial/\partial y_i$. Deduce that, whilst a $C^\infty(X)$-module it is <u>not</u> the space of sections of a vector bundle unless it has codimension 1 (a zero I suffices).

Now, given $m \in \mathbb{R}$ we define

$$I_{\mathcal{V}} H^m(X) = \{u \in H^m(X);\ \mathcal{V}^j u \in H^m(X)\ \forall j,$$
$$\mathcal{V} = \mathcal{V}(X;Y)\}$$

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These are the spaces we want, but the filtration is wrong! Not for wrong reasons gone. That is what we need to sort out.

The Fourier transform can be applied on the fibres of a real vector bundle to give an isomorphism

$$(FT) \qquad \mathcal{F}_{fib} : \begin{cases} \mathcal{S}'(V) \longrightarrow \mathcal{S}'(V', \Omega_{fib}). \\ \mathcal{S}(V) \longrightarrow \mathcal{S}(V', \Omega_{fib}) \end{cases}$$

Here Ω_{fib} is the space of fibre densities on V' and

$$\mathcal{F}_{fib} f = \hat{f}(x, w) = \int e^{-i w.v} f(x, v) |dv|$$

in terms of a local trivialization. In fact everything here is well-defined except for $|dv|$.

Exercise Check this claim, that if $\hat{f}\,|du|$ is taken as a fibre density on V' then it is completely well-defined, given (FT).

Proposition For any embedded submanifold

$$C^\infty(X, Y) \subset I_V H^\mu(X; Y) \subset C^\infty(X \setminus Y)$$

and if $\chi \in C^\infty(X)$ has support in a coordinate neighbourhood of Y then $\exists\, M \in \mathbb{R}$ s.t. $\mathcal{F}_{fib}(\chi u) \in \mathcal{S}^M(V') \otimes \Omega_{fib}$

and conversely $\exists\, M' < M$ s.t.

$$\chi\left(\mathcal{F}_{\mathrm{loc}}^{-1} S^{M'}(V') \otimes \Omega_{\mathrm{fcs}} \right) \subset I_V H^m(X;Y);$$

in fact we can take any $M' < -M - \frac{d}{2}$, $d = \dim Y$, and any $M \geq -m - d/2$.

Recall that $S^M(V')$ is the space of symbols on V', in local coordinates and trivialization (x,ξ)

$$a \in S^M(V') \iff |D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{M-|\beta|} \quad \forall \alpha,\beta.$$

<u>Exercise</u>: Try to check the proof of Lemma – I do not actually use it below. Just see that in local coordinates $u \in I_V H^m(E)$ satisfies

$$\partial_y^\alpha z^\gamma D_z^\beta u \in H^M \quad \forall \alpha,\beta,\gamma \text{ with } |\gamma| \geq |\beta|.$$

Under Fourier transform this turns into the estimate

$$\partial_y^\alpha D_\xi^\gamma \xi^\beta a \in (1+|\xi|)^M L^2 \quad \forall \alpha,\beta,\gamma, \ |\gamma| \geq |\beta|.$$

If you go back to an earlier lecture I think you will find I showed that this implies $a \in S^M$ and conversely $a \in S^{M'}$ implies this estimate.

<u>Definition</u> If $Y \subset X$ is embedded and x has support in the image of a normal fibration F we define

$$(I) \qquad I^m(X;Y) = F^* x \, \alpha^{y^{-1}}_{fb} \, S^{m-\frac{n}{2}+\frac{c}{4}}(V'; \Omega_{fb}) + C^\infty(X).$$

$n = \dim Y$, $c = \operatorname{codim} Y = n-d$.

This definition is bogus until we show that the left side is independent of the choice of F (and x but that is actually easy). This amounts to the coordinate-invariance of $I^m(X;Y)$ which is not so obvious since the Fourier transform of an L^∞-based space is involved. The transform of an L^∞-based space is involved.

From the discussion above, $\forall \varepsilon > 0$

$$(EF) \quad \begin{cases} I^m(X;Y) \subset I_V H^{-m+\frac{n}{2}-\frac{c}{4}-\frac{d}{2}-\varepsilon}(X;Y) \\ I_V H^{-m+\frac{n}{2}-\frac{c}{4}-\frac{d}{2}}(X;Y) \subset I^m(X;Y). \end{cases}$$

On the other hand the $I_V H^m$ spaces are manifestly coordinate-invariant so we just need to show:

<u>Lemma</u> Given a 1-parameter family F_t, $t \in [0,1]$, of normal fibrations of Y and $u = F_0^* x \, \alpha^{y^{-1}}_{fb} \, a$, $a \in S^{m-\frac{n}{2}+\frac{c}{4}}$

there is a smooth family $a_t \in S^{m-\frac{n}{2}+\frac{\varepsilon}{4}}$ such that

(F) $$\frac{d}{dt} F_t^* x \mathcal{L}_{fib}^{-1} a_t \in C^\infty([0,1]; I_V H^N(X;Y))$$

for any fixed integer N and $a_t - a_0 \in S^{m-\frac{n}{2}+\frac{\varepsilon}{4}-1}$ i.e.

<u>Proof</u> The derivative of such a family is given by the unique vector field V_t s.t. $\frac{d}{dt} F_t^* f = F_t^* V_t f$,

namely,

$$\frac{d}{dt} F_t^* x \mathcal{L}_{fib}^{-1} a_t = F_t^* \left(V_t x \mathcal{L}_{fib}^{-1} a_t + x \mathcal{L}_{fib}^{-1} \frac{da_t}{dt} \right)$$

Now, observe that the condition $(1,4)$ is a homogeneous Sobolev condition implying that V_t is actually of the form

(V) $$\sum_{fib} g_j W_t^j, \quad g_j \in C^\infty(X), \; g_j = O(x^2),$$
$$W_t^j \in \mathcal{V}(X;Y)$$

<u>Exercise</u> Check (V) by working in local coordinates (E).

Now from what we have seen before,

$$\mathcal{L}_{fib}^{-1} V_t x \mathcal{L}_{fib}^{-1} : S^m \longrightarrow S^{m-1}, \quad \forall m$$

Thus, just taking a constant in t already gives

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$$\frac{d}{dt} F_t^* \chi \alpha_{fb}^{-1} a \in I^{m-1} \quad \text{at least formally.}$$

To do this properly, define a_t iterative (to any finite order). Take $a_0^{(0)} = \text{const}$,

$$a_t^{(1)} = \int_0^t \frac{d}{dt} \alpha_{fb}^{-1} V_t \chi \alpha_{fb}^{-1} a \in C^\infty([0,1]; S^\mu)$$

$$\mu = m - \tfrac{n}{2} + \tfrac{c}{4},$$

$$= B a^{(0)}$$

and then $a_t^{(2)} = B a_t^{(1)}$ etc. For N large we arrive at (F). ∎

Then shows that (I) is right inverse to the symbol of F. From now, it gives us a complete natural short exact sequence

$$I^{m-1}(X;Y) \hookrightarrow I^m(X;Y) \xrightarrow{\sigma_m} \frac{S^{m-\frac{n}{2}+\frac{c}{4}}(N^*Y; \Omega_{fb})}{S^{m-\frac{n}{2}+\frac{c}{4}-1}}$$

<u>Exercise</u> Recall that the <u>classical</u> symbol

$$S_{cl}^\mu \subset S^\mu$$

$$(\Psi_b) \qquad \Psi_b^m(X;E,F) = \{A \in I^m(X^2_b; A_b, \mathrm{Hom}(E\times F)\otimes \pi_{R_b}^*\Omega_b,$$

$$A \equiv 0 \text{ at old boundary}\}.$$

Of course this second definition requires some comment. I have set things up so that the definition of $I^m(X;Y)$ works just as well in case of Y an <u>interior</u> p-submanifold of a compact manifold with corners. We already know that A_b is small, as it does not meet the 'old boundaries' of X^2_b, so (Ψ_b) does make sense, and is consistent with $\Psi_b^{-\infty}$. Note that the weird orders have disappeared since Δ_b has codimension n when X^2_b has dimension $2n$. [Except I messed up the definition!]

Now we just have to check that things work as advertised.

In particular, $I^m(X;Y)$ is a $C^\infty(X)$-module, with $\sigma(fu) = f|_Y \cdot \sigma(u)$. This allows us to extend the definition to sections of vector bundles with no effort.

$$I^m(X;Y,E) = I^m(X;Y) \otimes_{C^\infty(X)} C^\infty(X;E)$$

and to get the same short exact sequence and 'action' formula:

$$I^{m-1}(X;E) \hookrightarrow I^m(X;E) \xrightarrow{\sigma_m} S^{m-\frac{n}{2}+\frac{c}{4}}(N^*Y;E)$$

$$P \in \mathrm{Diff}^l(X;E,F) \to P: I^m(X;Y,E) \to I^{m+l}(X;Y,F)$$

$$\sigma_{m+l}(Pu) = \sigma(P)|_{N^*Y} \cdot \sigma(u)$$

<u>Definition</u> For a compact manifold without boundary

$$\Psi^m(X;E,F) = \{A \in I^m(X^2;\Delta, \mathrm{Hom}(E,F)) \otimes \pi_R^*\Omega\}$$

and for a compact manifold <u>with</u> boundary

as defined in terms of the radial compactification $\bar{V}$ of the vector bundle and ρ a defining function for the boundary by

$$S^M_d = \rho^{-M} C^\infty(\bar{V}).$$

Show that $I^m_d \subset I^m$, for any $Y \subset X$, given by the classical symbols is also well-defined.

<u>Proposition</u> If P is a differential operator on X, $P \in \mathrm{Diff}^k(X)$, then

$$P : I^m(X;Y) \longrightarrow I^{m+k}(X;Y) \quad \forall m \in \mathbb{R}$$

(PA)

$$\text{and} \quad \sigma_{m+k}(Pu) = \sigma_k(P)\big|_{N^*Y} \cdot \sigma_m(u).$$

<u>Proof</u>. It suffices to check this for vector fields and their iterates. From the coordinate-independence and the fact that everything is well-defined, it is enough to work locally. Then $Y = \{z_i = 0\}$ and

$$P = V = \sum_i a_i(y,z)\,\partial_{y_i} + \sum_j b_j(z,y)\,\partial_{z_j}$$

In terms of (F), the first part is in $V(X;Y)$ and maps I^m to I^m. The second gives (PA) easily.