Conormal distributions

The aim today is to properly introduce the space \( \mathbb{S}^k_0(\mathbb{R}) \) for \( k \in \mathbb{R} \) and describe as many of its properties as I can get to. A little bit of standard geometry to start with.

A closed subset \( \gamma \subset X \) is an embedded submanifold in a compact manifold (for the moment without boundary) if \( C^\infty(\gamma) \cong C^\infty(X) \). This is equivalent to the existence at each point \( \gamma \) of local coordinates

\[
y_1, \ldots, y_p, s_1, \ldots, s_{n-p} \subset \gamma \subset X
\]

such that \( \gamma \cap U = \{ s_i = 0 \} \).

Theorem [Closed neighborhood] For any embedded submanifold \( \gamma \subset X \) without boundary, there is an open neighborhood \( \gamma \subset U \subset X \) at \( \gamma \) with \( C^\infty(\gamma) \cong C^\infty(U) \), if the zero section of the normal bundle at a differential.

\[ F : U \supseteq \rightarrow U' \]

and

1. \( F(y) = y \) for \( y \in \gamma \), \( F : \gamma \rightarrow \gamma \)

2. \( F_x : \gamma \rightarrow \gamma \) for \( x \in \gamma \)
and any such maps as bounds through a small family will have property <

Note that $N_y Y = TyX/TyX$ and for $y \in Q \subseteq Ty(N_y X) = N_y Y \equiv N_y Qy$ since the fibre is cotangent to the zero section.

Part I am not going to do here in detail since it is very standard. The clearest proof I know of uses a Riemann metric $a X$ at fixed $F$ through the exponential map. The Riemann metric allows $N_y Y$ to be identified with the null component $(TyX)^+ \cap TyX$ and then one can check readily that

$$F(y, v) = \exp_y (v)$$

has the desired properties. Then $F(y, v)$ will be the point of parameter distance one along the geodesic with initial point $y$ and initial tangent vector $v$, so if you prefer to obtain $1/4$ for the geodesic with initial vector $v/\|v\|$.

The 'uniqueness' part of this end can be based on the existence of such a 'normal fibration' for the diagonal $\Delta$.\]
In the case of a submanifold $Y \subset X$ of a compact
manifold $X$, it comes we assume that $Y$ is a pseudo-submanifold
(think $p$- for product). This is the condition that
new each $y \in Y$ there an local coordinate, at the usual
adapted sort, $\xi_j \to x_1, \xi_j \to x_2, \ldots\). Which it looks at
what $Y$ is locally

(P): $x_1 = \ldots = x_j = 0, y_1 = \ldots = y_{p-j} = 0$.

Usually we demand $j = 0$, in what case $Y$ is an
integral pseudo-submanifold (attain not to an integral
pseudo-submanifold on some boundary face).

Exercise: Note that if we don't assume (P) explicitly but just require that $C^0(X)/Y = C^0(Y)$ be
the $C^0$-sections we allow things like $x = x'$ is

\[ \begin{array}{c}
\begin{array}{|c|}
\hline
x \\
\hline
y
\end{array}
\end{array} \]

Show how to recover (P) locally by assuming just
that $C^0(X)/Y = C^0(Y)$ and some condition of $T_p X$ for
The collar neighborhood $B_n$ goes through unchanged for a $p$-submanifold of manifolds with corners compact.

Forget the boundary core for the moment, this is not the important part.

We introduce spaces of conormal distributions associated to an embedded submanifold $Y \subset X$, then as distributions in $X$, singular only at $Y$. Put

$$\mathcal{D}(X, Y) = \left\{ V \in C^\infty \text{ vector fields on } X, \text{ target } Y \right\}$$

Exercise show that in local coordinates $(E)$, $\mathcal{D}(X, Y)$ is spanned by $\partial_j$, $\partial_s$, $\partial_t$. Deduce that, when a $C^\infty(X)$-module, not as with the space of sections of a vector bundle unless it has codimension 1 (a fact I suppose).

Now, given $M \subset \mathbb{R}^n$ we define

$$I_n H^m(X) = \left\{ u \in H^m(X); \forall j \left( \forall t \in (X, \forall y) \right)$$

$$\left. u^{(j)} \right|_{y} = 0 \right\}$$
There are the spaces we need, but the filtration is wrong!

Not for every mind you. That is what we need to sort out.

The Fourier transform can be applied to the fibres of a real vector bundle to give an isomorphism

\[
\hat{\mathcal{F}}: \mathcal{L}'(V) \rightarrow \mathcal{L}'(V, \Omega^\bullet_{\mathcal{F}^\bullet}). \\
\mathcal{L}(V) \rightarrow \mathcal{L}(V', \Omega^\bullet_{\mathcal{F}^\bullet}).
\]

Here \( \Omega^\bullet_{\mathcal{F}^\bullet} \) is the space of fibre densities on \( V' \) and

\[
\hat{\mathcal{F}} f = \hat{f}(x, \omega) = \int e^{-i\omega \cdot x} f(x, \nu) d\nu
\]

is true of a local trivialisation. In fact every thing has a well-defined except for \( d\nu \).

Examine that the claim, that if \( f d\nu \) is taken as a fibre density on \( V' \) then it is completely well-defined up to gauge (F1).

**Remark.** For any embedded submanifold \( Y \)

\[
C^0(Y, Y) \subset \mathcal{H}(X; Y) \subset C^0(X \times Y)
\]

and if \( x \in C^0(X) \) has support in a cell neighbourhood of \( Y \) then \( \forall \epsilon > 0 \) s.t. \( y(x) \epsilon \mathcal{S}(V) \otimes \Omega^\bullet_{\mathcal{F}^\bullet} \)
at every $T \in C^0[0,1]$

$$X(\mathcal{L}^d, S^m(\Omega) \otimes \mathcal{D}[\mathcal{E}]) \subset \mathcal{D}[\mathcal{E}].$$

It is in fact an easy task to any $M < -m - \frac{d}{2}$, $d = \text{dim} \Omega$, and any $M \geq -m - \frac{d}{2}$.

Recall that $S^m(\Omega)$ is the space of regular on $\Omega$ polynomials of degree at most $m$.

$$\mathcal{L}^d S^m(\Omega) \subset \mathcal{D}[\mathcal{E}].$$

Let us define

$$u \in \mathcal{D}[\mathcal{E}] \quad \text{such that} \quad \frac{1}{k} \mathbf{D}^k u \leq C_k \mathbf{D}^k (e^{\lambda \mathbf{D}^2} \phi_{\mathbf{D}^2}).$$

Exercise: Try to check that of these. I do not actually use it below. Just see that it is local cohomology.

$$H^m \subset \mathcal{D}[\mathcal{E}].$$

Under Fourier transform, then the rule is the action

$$\mathbf{D}^k u \in H^m \quad \forall \mathbf{D} \leq 1 \mathbf{D}^2$$

write $\phi_{\mathbf{D}^2}$

If you go back to an earlier lecture I think you will find I showed that the right of $S^m$ does converge at $S^m$ and that this estimate is true.
Definition If \( Y \subset X \) be embedded at \( x \) by \( \phi \), we define

\[
I^m(x; y) = F^* \left( \frac{y}{\phi} \right)^{m-n-\frac{c}{2}} S^{m-n-\frac{d}{2}} (V; P_{\phi})^c + C^a(x).
\]

\( m = \text{dim } X, c = \text{codim } Y = n - d. \)

The definition is Bogomolov as we show that the left side is independent of the choice of \( F \) (at \( x \) but not \( d \) locally). This amounts to the coordinates-invariant form \( I^m(x; y) \) which is not so obvious since the forms of the form of an \( L^m \)-based space are involved. For

From the discussion above, \( \forall \infty \)

\[
I^m(x; y) \leq I^m(x; y) \leq I^m(x; y).
\]

\( (EF) \)

\[
I^m(x; y) \leq I^m(x; y) \leq I^m(x; y).
\]

On the other hand, the \( I \) \( H^m \) spaces are manifestly coordinate-invariant so we just need to show

Lemma Given a \( k \)-parametric family \( F_t \) at bound

fibration of \( Y \) and \( n = F_0^* \times \frac{y}{\phi} \), at \( S \).
Here is a smooth family of $F \in S$ such

\[
(F) \quad \frac{d}{dt} F^x \times \delta^{-1} \cdot F^x \in C^1([0,1]; H^N(x;\gamma))
\]

for any preassign $N \geq q - q \in S$.

Now, observe that the condition $V_t$ as a vector field.

Now, consider that $V_t$ is actually a vector field.

\[
(V) \quad \sum_{j=1}^{\infty} \gamma_j W_t \quad \text{with} \quad \gamma_j \in C^1(x;\gamma), \quad \gamma_j = 0 \quad \forall t,
\]

\[
W_0 \in V(x;\gamma)
\]

Each $\gamma$ of $(V)$ by working in local coordinates $(E)$. Next, from what we have seen before,

\[
V_t \times \delta^{-1} : S^n \to S^{n-1}
\]

Thus, just taking a counter not already given.
$$\frac{1}{t} F^* x \otimes \phi, \alpha \in \mathcal{H}^{\infty-1} \text{ at best formally.}$$

To do this path, define \( \alpha \) to vanish at \( t \) any finite order. Take \( \alpha^{(w)} = \text{const} \),

$$a^{(w)}_t = \int_0^t \frac{1}{y} \frac{\partial}{\partial t} x \otimes \phi, \alpha \in \mathcal{H}^{\infty-1} \text{ at } t = (\alpha_{u(t), \phi}^{(w)}),$$

$$N = \mu - \frac{n - \frac{n - \epsilon}{4}}{4},$$

at \( t \), \( a^{(w)}_t = Ba^{(w)}_t \) et cetera. Fix \( N \) large enough.

This shows that (1) is interpretable at the claim of \( F \). Even more, it gives us a complete notion

**short exact sequence**

$$I^{\infty-1}(x; y) \rightarrow I^{\infty}(x; y) \rightarrow S^{m-n+\frac{n-\epsilon}{4}} \rightarrow S^{m-n+\frac{n-\epsilon}{4}-1}$$

**Exactness Recall 1st k-classical symbol**

$$S^m \subset S^4$$
\[ \Phi^n_{(x, E, F)} = \{ A \in \mathcal{D}^n (x_b^1; A_b), \text{tr}(E,F) \leq b \} \]

\[ A = 0 \text{ at all boundary } \gamma. \]

Of course the second definition remains some constant. I have set things up so that the definition of \( I(x^y) \) looks just as well as one of \( J \) an integer prescribed at a compact manifold with corners. We already know that \( A_b \) as such, as it does not meet the \( \text{old boundary} \) at \( x^b \), so \( (y_b) \) also works. I don't recall what mentioned with \( \Phi^n_{(x^b)} \). Note that the word order has disappeared since \( A_b \) has codimension \( n \) when \( x^b \) has dimension \( 2n \). [Except I messed up the definition!]

Now we just have to check that things work as advertised.
In particular, $I^u(x, y) \to \mathcal{C}^0(X)$ is finite, with $\sigma(f, u) = f/y \cdot (\nu)$. The above can be extended via the definition in section 4 under weaker and softer effects.

$$I^u(x, y, E) = I^u(x, y) \otimes \mathcal{C}^0(X; E)$$

and to get the same short exact sequence as that earlier found:

$$0 \rightarrow I^u(x, E) \rightarrow I^u(x; E) \rightarrow S^{(\nu)}(x; E)$$

$$\begin{array}{c}
0 \rightarrow \text{Diff}^1(x; E, F) \rightarrow P: I^u(x, E) \rightarrow I^u(x, F)
\end{array}$$

$$\sigma_{m+1}(P, u) = \sigma(P)/\nu y$$

**Definition.** For a compact manifold without boundary

$$\mathcal{D}^u(x, E, F) = \{ A \in I^u(x; A, \text{Hom}(E, F)) \}$$

and for a compact manifold with boundary
a definite integral of the radial component

\[ V \] of the vector field and \( p \) a density function for \( X \)

leading to

\[ S_d = p^{-1} c^0(V). \]

Show that \( I^m c I^m, \) for any \( Y \subset X, \) gives by integrals over cells well-defined.

Proposition: If \( P \) is a differential form \( \alpha, \) \( \text{PDEff} \) (x)

Let

\[ P : I^m(x; y) \to I^{m+1}(x; y) \quad \text{then} \quad R \]

\[ \sigma(Pu) = \sigma_k(P) \frac{1}{N^y} \cdot \sigma_m(u), \]

Proof. It suffices to check this for vector fields \( \alpha \)

and their representatives for the coordinates - independent of

the fact that everything is well-defined, it is enough
to each coordinate. Thus \( y = \mathbf{3} = 0 \) ad

\[ P = V = \sum a_i(g, \mathbf{3}) \partial y_i + \sum b_i(g, \mathbf{3}) \partial y_j \]

of \( \mathcal{F} \). the first part is \( V(x; y) \) ad

with \( I^m \) to \( I^m \). The second part \( (PDEff) \) and