Last time I described, for a comic metric, the space
\[ D = \{ u \in L^2 (\mathbb{R}; \Lambda^*) ; \text{d}u, \text{d}u \in L^\infty (\mathbb{R}; \Lambda^*) \} . \]
It consists of four pieces. The largest is simply the closure of \( C^\infty (\mathbb{R}; \Lambda^*) \) in the Hilbert space \( x^{-\frac{n}{2}+1} t^{1/6} L^1 (\mathbb{R}; \Lambda^*) \). With respect to the norm
\[
\| u \|_D^2 = \| u^2 \|_{L^1 (\mathbb{R}; \Lambda^*)} + \| \text{d}u \|_{L^\infty (\mathbb{R}; \Lambda^*)} + \| \text{d}u \|_{L^2 (\mathbb{R}; \Lambda^*)} .
\]
Apart from these three finite-dimensional pieces, the associated boundary cohomology
\[
\left\{ \begin{array}{ll}
\{ & x \cdot \text{H}^{\frac{n}{2}-\frac{1}{2}} \text{Ho}(\mathcal{A}) (u) \\
\{ & x \cdot dx \wedge \text{H}^{\frac{n}{2}+\frac{1}{2}} \text{Ho}(\mathcal{A}) (u) \\
\end{array} \right.
\]
what only exist for \( u \neq 0 \) and
a 'non-cohomology' for $G$. It shows, if we define briefly, that $G \subset D_A \cap D_R$, we can be appropriately approximate.

We will now show that $E_A \cap D_R = \{0\}$, which will complete the description of $D_R$ and hence $D_A$. To do this we compute directly the quadratic form

$$Q(u, v) = \int \left( \langle u, v \rangle - \langle u_1, v_1 \rangle \right) \, dy,$$

$\quad \mu_{\text{red}}.$

Observe that this vanishes on $D_R$, since if $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then $Q(u, v) = 0$.

Lemma. If $u = x \varphi \in E_{\mu}$ and $v = x \varphi \in E_{\mu}^{\frac{n}{2} - \frac{1}{2}}$, then

$$Q(u, v) = \int \langle \varphi, \varphi \rangle \, dx.$$
Proof: For all $x \in \mathbb{R}^\frac{n+1}{2}$, $d\nu = 0$ at $x \in \mathbb{E}_A$, $d
u = 0$. Thus,

$$\Phi(x) = \int \langle d(u, v) \rangle d\mu$$

$$= \int \int 2xx' \langle \Phi, \psi \rangle \, dx \, dh_0$$

$$= \int \langle \Phi, \psi \rangle \, dh_0.$$ 

From here it follows that $\mathbb{E} \cap \mathbb{E} = \emptyset$.

Since $\mathbb{E}^{\frac{n+1}{2}} \subset \mathbb{D}$,

Now we are in a position to form the main proposition leading to the Hodge decomposition, namely that $d + \delta$ is self-adjoint and Fredholm on $\mathbb{D}_A$ and $\mathbb{D}_B$.

For self-adjointness, recall that we define...
\( D^*_R = \{ u \in L^2 : D \varphi \mapsto \langle (d+\delta) \varphi, u \rangle \text{ extends by continuity to } L^2 \} \).

Recall that \( D^*_R \) is graded by degree so we see that \( u \in D^*_R \) implies that each form component \( u_{(h)} \in D^*_R \) and

\[
D^*_R \varphi \mapsto \langle d \varphi, u \rangle = \langle \varphi, u \rangle
\]

\[
D^*_R \varphi \mapsto \langle d \varphi, u \rangle = \langle \varphi, u \rangle
\]

but extends by continuity to \( L^2 \) — simply restrict to the af-tributed form degree \( k+1 \).

Thus, \( u \in D \) since \( d u, d \varphi \in L^2 \).

\[ D^*_R \subset D. \]

It is also clear that \( D^*_R < D^*_R \) since if \( u \in D^*_R \),

\[
\langle (d+\delta) \varphi, u \rangle = \langle d \varphi, u \rangle + \langle d \delta \varphi, u \rangle
\]

\[ = \lim_{n \to \infty} \left( \langle d \varphi, u \rangle + \langle d \delta \varphi, u \rangle \right) \]

\[ = \lim_{n \to \infty} \left( \langle d \varphi, u \rangle + \langle \varphi, d u \rangle \right) = \langle \varphi, d \delta \varphi \rangle. \]
using the approximations built to $\varphi$ ad $n$.

This we only need show that $E_\Lambda \cap D^n_R = \{0\}$
and this is the same argument as before.

Namely, essentially by definition,

$$Q(\varphi; u) = ((d + S)\varphi, u) - (\varphi; (d + S)u)$$

$$= 0 \text{ on } D^n_R \times D^n_R.$$  

Taking $\varphi \in E^n_R$, it follows that $E^n_\Lambda \cap D^n_R = \{0\}.$

Finally, let us check the Fredholm property for $d + S$ on $D_R$, say.

The main point here is that $D$ (or $D_R$) with the norm $(\|D\|)$ induce compactness in $L^2$.

(5) $I: D \rightarrow L^2_g, \quad I(B) \subset L^2_g$ is compact if $B$ is compact,

Then as $L^2$ Ascoli-Arzela-Stein, namely

$$D \subset x^{-\frac{1}{2}}H^1(D, x^1)^\perp \subset \frac{\alpha + \theta}{2}L^2(D, x^1).$$
and the delta already rejects company with $L^2$.  

Exercise: Check this!

From this compactness one deduce immediately that

$$H^* (X) = \{ u \in D' \mid (d+\delta)u = 0 \}$$

is finite dimensional, since it is closed in $L^2$. (by the compact content of $d+\delta$'s distributions) and has compact support ball.

Similarly, the range

$$(d+\delta) D' \subset L^2$$

is closed.

Indeed, if $(d+\delta) u_n \to u$ in $L^2$ then we can assume $u_n \in H^* (X)$. The coefficients $d \delta u_n, u \in D'$ show $d u_n \to u, \delta u_n \to u$, $u \subset u + u_\delta$. 
If we go back to the derivation of the structure of $D$ we can apply the same argument, first concluding that $u_h \to u$ in $\mathcal{A} L_2^{1/2} H_{0}^{1}(\mathcal{X}, \mathcal{V})$, using elliptic regularity. From this we deduce

$$ (d + S_0) u_h \to (d + S_0) u \in L^2_{\mathcal{X}} $$

and hence

$$ u_h \to u \text{ in } D (\text{at least } D_R), $$

using the Mellin transform.

Finally then we have most of what we set out to get for its cores — Hodge decomposition and identification of Hodge and $L^2$ cohomology, subject however to elliptic regularity (which despite my delaying the proof, is not supposed to be hard!). We still need to check the identification of it with cohomology, but I will get back to that.

So, back to more geometric analysis.
If you recall I had introduced a space $X^0_b$ by turning up the corner in $X^2$, where $X$ is a (compact) manifold with boundary. If we go back to the beginning when we thought a little about identifying $X$ with or from $C^\infty(X)$. From this point of view we can define

$$C^\infty(X^0_b) = \{ u \in C^\infty(X^2 \setminus \partial X^2) : \text{u is } C^\infty \text{ in polar coordinates around } \partial X^2 = \{ x = x_b = 0 \} \}.$$ 

Of course we still have to say all polar coordinates or show that the coordinates is independent of which polar coordinates (that is, which coordinates we take before we reside dual polar coordinates). I already dealt this.

Namely, if we just take a compact
manifold not lying at focuses on a particular branch from $F_i$ of codimension $k$, we are reduced to the following

**Lemma.** If $F: U, 0 \rightarrow \mathbb{R}^{n-k}$ is a differentiable map from a neighborhood $U$ of $0 \in \mathbb{R}^{n-k}$ into a neighborhood of $0$ then $M \subset \mathbb{R}^n$ is $z_1 + \cdots + z_k$, $\frac{a_i - z_i+1}{a_i + \cdots + z_k}$, $i=1, \ldots, k-1$

at $y_1, \ldots, y_k$ which replaces the same restriction of $F^*_n$.

**Proof.** It suffices to show that the Jacobian coordinates function $a_i + \cdots + z_k = r$ at $\xi = (a_i - z_i+1)/r$ pull back to $\mathcal{C}^\infty$ functions (of the coordinates). For it, $y_i$'s the coordinate. Reversing the coordinate change we see that

\[ a_i - z_i+1 = t_i r, \quad i=1, \ldots, k-1 \]

\[ \Rightarrow \quad z_i = (t_i r), \quad t_i = t_i + t_i' \]
For vector $v$, what I will teach you to evaluate, haven't do form a basis of $\mathbb{R}^k$, with $\ell_i \geq 0$ of all $i \in \{1, \ldots, k\}$.

By assumption, $F$ preserves $\mathbb{R}^k$, locally.

Let's assume for simplicity that

$$F^* x_j = a_j x_j, \quad 0 < a_j \in \mathbb{C}$$

Since this must be true if $F$ is rearranged.

Thus

$$F^* r = \sum_j (a_j \ell_j) \cdot r = a r,$$

$$0 < a \in \mathbb{C}.$$

When I have you to check this step.

From this it follows that

$$F^* \bar{t}_i = \frac{F \bar{x}_i - F \bar{x}_{i+1}}{\alpha r}$$

$$= \frac{1}{\alpha}(a_i \ell_i + a_{i+1} \ell_{i+1}(t_i))$$

to $\mathbb{C}^k$ as claimed.
Then we know that $[X; F]$ is a well-defined compact manifold with corners when

$$C^0([X; F]) = \left\{ u \in C^0(X \setminus F) \mid u \text{ is } C^0 \text{ if polar cords near each } f \in F \right\}$$

It is important to note that $[X; F]$ can be written as a smooth blow-down in $\mathbb{R}$

$$\beta: [X; F] \to X$$

at $f \in F$, as a set,

$$X \setminus F \cup F \times [1, 2]_{k-1, k-1}$$