DIRAC OPERATORS AND BOUNDARIES, 18.158, FALL 2003

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Abstract. Notes from my lectures in Fall 2003 in MIT course 18.158.

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Introduction

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$$n \text{ even } k = \frac{n}{2},\ 0 < e_j \leq 1. \tag{1}$$

The poles of $u^d_{t,M}$ can only be in the same places.

9. Lecture IX, October 7, 2003

Let me start today by doing a little piece of analysis using the Mellin transform.

Lemma 1. *If $u \in x^{-t}L^2_{\mathrm{b}}([0,\infty))$ for some $t > 0$ has support in $x < 1$ and is such that $x\frac{d}{dx}u \in L^2_{\mathrm{b}}([0,\infty))$ then there exists $u_j \in \dot{\mathcal{C}}^\infty([0,\infty))$ such that $u_j \to u$ in $x^{-t}L^2_{\mathrm{b}}([0,\infty)$ and $x\frac{d}{dx}u_j \to x\frac{d}{dx}u_j$ in $L^2_{\mathrm{b}}([0,\infty)$.*

Exercise 1. If you are so inclined, find a proof which does not use the Mellin transform!

Proof. First note that if $u \in L_{\mathrm{b}}(X)$ then $x^\epsilon u \to u$ in $L^2_{\mathrm{b}}(X)$ as $\epsilon \downarrow 0$ for any compact manifold with boundary.

Exercise 2. Write out a careful proof of this.

Since we know that $x\frac{d}{dx} : H^1_{\mathrm{b}}([0,\infty) \longrightarrow L^2_{\mathrm{b}}([0,\infty))$ is continuous and that $\dot{\mathcal{C}}^\infty([0,\infty))$ is dense in $H^1_{\mathrm{b}}([0,\infty))$ it suffices to show that the sequence can be chosen in this space. So, the obvious way to get such a sequence is to take $u_j = x^{\epsilon_j}u$, with $\epsilon_j \downarrow 0$. From the Paley-Wiener theorem, the Mellin transform of u is holomorphic in $\operatorname{Im} s < -t$ and is square integrable on real lines in this half space with uniformly bounded L^2 norm. On the other hand,

$$(x\frac{d}{dx}u)_M = -isu_M \tag{1}$$

must be similarly holomorphic and L^2 in $\operatorname{Im} s < 0$. Thus certainly, $u_j \to u$ in $x^{-\delta}L^2_{\mathrm{b}}([0,\infty)$ for any fixed $\delta > 0$ (and in particular u lies in this space.) Now

$$x\frac{d}{dx}(x^\epsilon u) = x^\epsilon(x\frac{d}{dx}u) + \epsilon x^\epsilon u$$

and as already noted, the first term converges in L^2_{b} to $x\frac{d}{dx}u$ as $\epsilon \downarrow 0$ so it suffices to show that the second term converges to 0 in this space. The square of the L^2 norm of its Mellin transform may be estimated as follows:

$$\epsilon^2 \int_{\mathbb{R}} |u(s-i\epsilon)|^2 ds$$
$$\leq \epsilon \int_{|s|\geq \epsilon^{\frac12}} |(s-i\epsilon)u(s-i\epsilon)|^2 ds + \int_{|s|\leq \epsilon^{\frac12}} |(s-i\epsilon)u(s-i\epsilon)|^2 ds$$

where in the first term the estimate $|s-i\epsilon|^2 \geq \epsilon$ is used and in the second $|s-i\epsilon| \geq \epsilon^2$. By the assumed square-integrability of $x\frac{d}{dx}u$ both terms tend to 0 with ϵ. □

Using this and some related analysis I next want to write down the domains of $d+\delta$ that we have been discussing. First, we always have

$$x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \subset D_A \cap D_R. \tag{2}$$

This is a direct result of the fact (discussed further below) that

$$\dot{\mathcal{C}}^\infty(X;\Lambda^*) \subset x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \tag{3}$$

is dense with respect to the natural Sobolev norm and

$$d, \delta : x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \longrightarrow L^2_{\mathrm{c}}(X : {}^{\mathcal{C}}\Lambda^*) \tag{4}$$

are continuous. Thus for an element $u \in x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ there is an approximating sequence $\phi_j \in \dot{\mathcal{C}}^\infty(X;\Lambda^*)$, with $\phi_j \to u$, $d\phi_j \to du$ and $\delta\phi_j \to \delta u$ all in $L^2_g(X;\Lambda^*)$.

From the behaviour of solutions to $du = 0$ and $\delta u = 0$ in the model case we can add a few more pieces to the domains. These are all determined by the eigenfunctions of the limiting metric, h_0, on the boundary. Choose $\chi \in \mathcal{C}^\infty(X)$, a cut-off function supported very near the boundary and identically equal to 1 in some neighbourhood of it. Then set, for n odd

$$\begin{aligned} E_A &= \chi \cdot H^{\frac{n}{2}-\frac{1}{2}}_{\mathrm{Ho}(h_0)}(\partial X), \\ E_R &= \chi \cdot dx \wedge H^{\frac{n}{2}-\frac{1}{2}}_{\mathrm{Ho}(h_0)}(\partial X) \end{aligned} \tag{5}$$

in terms of the Hodge cohomology, i.e. harmonic forms, on the boundary for the metric h_0.

Similarly we fix spaces associated to non-harmonic eigenforms of the tangential Laplacian. If λ is such an eigenvalue for exact k forms on the boundary, so there is a non-trivial

$$0 \neq e_\lambda \in \mathcal{C}^\infty(\partial X; \Lambda^k),\ e_\lambda = de'_\lambda,\ d\delta e_\lambda = \lambda e_\lambda \tag{6}$$

we consider

$$f_\lambda = x^{-is_\lambda}(x^k(-is_\lambda + k)e_\lambda + x^{k-1}dx \wedge e'_\lambda),\ \text{where} \tag{7}$$
$$-is_\lambda = -\frac{n}{2} + |\frac{n}{2} - k| + e_\lambda,\ e_\lambda = \sqrt{(\frac{n}{2} - k)^2 + \lambda} - |\frac{n}{2} - k|$$

and then let

$$\begin{aligned} G^{\frac{n}{2}-\frac{1}{2}} &= \sum_{0<e_\lambda<\frac{1}{2},\ k=\frac{n}{2}-\frac{1}{2}} \mathbb{C}\chi f_\lambda, \\ G^{\frac{n}{2}} &= \sum_{0<e_\lambda<1,\ k=\frac{n}{2}} \mathbb{C}\chi f_\lambda, \\ G^{\frac{n}{2}+\frac{1}{2}} &= \sum_{0<e_\lambda<\frac{1}{2},\ k=\frac{n}{2}+\frac{1}{2}} \mathbb{C}\chi f_\lambda \end{aligned} \tag{8}$$

be the corresponding finite dimensional subspace of k-forms on X. Here, each eigenvalue of the boundary Laplacian on exact k forms, with e_λ in the indicated range, is repeated with its (finite) multiplicity, as the e_λ run over a basis. Of course the first and third spaces only make sense when n is odd, and the second when n is even.

Observe that each of these spaces is contained in $L^2_g(X; \Lambda^*)$ but intersects the smaller space $xL^2_g(X; \Lambda)$ in 0. The point here is that

$$du,\ \delta_0 u \in \dot{\mathcal{C}}^\infty(X; \Lambda^*),\ u \in G^k \tag{9}$$

provided δ_0 corresponds to a product-type conic metric, equal to $dx^2 + x^2 h_0$ near the boundary.

Exercise 3. Check (9) carefully! It follows from the formulæ for d and δ and the fact that the 2-vector implicit in (7) is a null vector of the 2×2 matrix implicit in the computation of the joint (formal) null space of d and δ_0 above

$$\begin{pmatrix} -1 & -is_\lambda + k \\ is_\lambda - (n-k) & \lambda \end{pmatrix} \begin{pmatrix} -is_\lambda + k \\ 1 \end{pmatrix} = 0. \tag{10}$$

Here of course s_λ has been chosen so the matrix has rank 1.

This takes care, as we shall see, of all the possible poles we discovered within the 'critical strip' $-\frac{n}{2} < \operatorname{Im} s < -\frac{n}{2} + 1$ for the Mellin transform of a form annihilated by d and δ_0. We need also to consider the poles on the line $\operatorname{Im} s = -\frac{n}{2} + 1$. To handle these we consider an infinite-dimensional space of functions on the line

$$\mathcal{L} = \left\{ h \in x^{-\epsilon} L^2_{\mathrm{b}}([0,\infty)), \epsilon > 0; h = 0 \text{ in } x > 1,\ x\frac{d}{dx}h \in L^2_{\mathrm{b}}([0,\infty)) \right\}. \tag{11}$$

Notice that Lemma 1 applies to elements of this space and shows in particular that it is independent of the choice of ϵ. With these functions as coefficients we consider spaces related to those in (5) and determined by the harmonic $\frac{n}{2} - 1$ forms on the boundary with respect to h_0 :

$$\begin{aligned} E^{\frac{n}{2}-1}_{\mathcal{L}} &= \mathcal{L}(x) \cdot H^{\frac{n}{2}-1}_{\mathrm{Ho}(h_0)}(\partial X), \\ E^{\frac{n}{2}+1}_{\mathcal{L}} &= x^2 \mathcal{L}(x) \cdot dx \wedge H^{\frac{n}{2}-1}_{\mathrm{Ho}(h_0)}(\partial X). \end{aligned} \tag{12}$$

Exercise 4. Again you should do the little computation to see that if n is even then

$$E^{\frac{n}{2}\pm 1}_{\mathcal{L}} \subset L^2_g(X; \Lambda^*) \text{ and } u \in E^{\frac{n}{2}\pm 1}_{\mathcal{L}} \Longrightarrow du,\ \delta_0 u \in L^2_g(X; \Lambda^*). \tag{13}$$

Similarly we consider spaces closely related to those in (8) involving the form (7) corresponding to an exact boundary k-form which is an eigenform for the boundary Laplacian:

$$\begin{aligned} G^{\frac{n}{2}-\frac{1}{2}}_{\mathcal{L}} &= \sum_{e_\lambda = \frac{1}{2},\ k = \frac{n}{2} - \frac{1}{2}} \mathcal{L} \cdot f_\lambda, \\ G^{\frac{n}{2}}_{\mathcal{L}} &= \sum_{e_\lambda = 1,\ k = \frac{n}{2}} \mathcal{L} \cdot f_\lambda, \\ G^{\frac{n}{2}+\frac{1}{2}}_{\mathcal{L}} &= \sum_{e_\lambda = \frac{1}{2},\ k = \frac{n}{2} + \frac{1}{2}} \mathcal{L} \cdot f_\lambda. \end{aligned} \tag{14}$$

Notice that the non-triviality of these spaces corresponds to an 'accident' in which there is a positive eigenvalue for which e_λ takes on a specific value.

Exercise 5. If you haven't thought about this already, given an example of a function which is in $\mathcal{L}$ but is not in $L^2_{\mathrm{b}}([0,\infty))$.

Finally we get to an explicit description of the domains.

Proposition 1. *For a conic metric on a compact manifold with boundary*

$$\begin{aligned} D &= \left\{ u \in L^2_g(X; \Lambda^*); du, \delta u \in L^2_g(X; \Lambda^*) \right\} \\ &= x^{-\frac{n}{2}+1} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) + E^*_A + E^*_R + G^* + E^*_{\mathcal{L}} + G^*_{\mathcal{L}}; \end{aligned} \tag{15}$$

and D_A and D_R are the same without the summands G^*_R and G^*_A respectively.

Remark 1. a) Before proceeding to the proof of this, note that the difference between D_A and D_R amounts to the repalacement of a finite dimensional subspace of the domain by another, of the same dimension – because by Poincaré duality $H^{\frac{n}{2}\pm\frac{1}{2}}(\partial X)$ have the same dimension.

b) The 'complicated' (in particular infinite-dimensional) extra terms in (15), $E^*_{\mathcal{L}}$ and $G^*_{\mathcal{L}}$, are really rather insignificant. As follows from the discussion below, if we give D the obvious norm

$$\|u\|^2_D = \|u\|^2_{L^2_g} + \|du\|^2_{L^2_g} + \|\delta u\|^2_{L^2_g} \tag{16}$$

then $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) + E^*_{\mathcal{L}} + G^*_{\mathcal{L}}$ is the closure of $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ (and hence also of $\dot{\mathcal{C}}^\infty(X; \Lambda^*)$) in D.

c) In particular this means that the quotient of D by D_0, the closure of $\dot{\mathcal{C}}^\infty(X; \Lambda^*)$ in D, is finite dimensional.

Exercise 6. Check the statement following (16); the discussion below shows that this set is contained in the closure; the converse amounts to the exclusion of the other sets E^*_A, E^*_R and G^*. For the first two this is done below and a similar argument also works for the third.

Exercise 7. Show that the bilinear form

$$W : D \times D \ni (u, v) \longmapsto \int_X ((du + \delta u, v) - (u, dv + \delta v))\, dg \tag{17}$$

is antisymmetric (if we are dealing with real forms and the real pairing) and vanishes on D_0. Let $D_m = D_A \cap D_R$ be the subspace of D consisting of the elements have approximating sequences $u_j \in \dot{\mathcal{C}}^\infty(X; \Lambda^*)$ such that $u_j \to u$ and $du_j \to du$ in L^2_g and also are approximable in L^2_g by a possibly different sequence v_j for which δv_j converges in L^2_g. Show that W vanishes on D_m and that that D/D_m is a symplectic vector space in which D_A/D_0 and D_R/D_0 are complementary Lagrangian subspaces.

Proof. From elliptic regularity (which I still have to prove) we 'know' that

$$u \in L^2_g(X; {}^{\mathcal{C}}\Lambda^*),\ du, \delta u \in L^2_g(X; {}^{\mathcal{C}}\Lambda^*) \Longrightarrow u \in x^{-\frac{n}{2}} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*). \tag{18}$$

Thus, we start off with one factor of x less than we need to get into the first term in the putative expansion of D.

Exercise 8. Check again that you know why all the terms in (15) are in $L^2_g(X; {}^{\mathcal{C}}\Lambda^*)$.

Now, we are dealing with a conic metric which is not necessarily of product type near the boundary. On the other hand, the result we are looking for only depends on the limiting metric h_0 and not the higher perturbations

$$g = dx^2 + x^2 h(x, y, dy, dx) = g_0 + xq(x, y, xdy, dx),\ g_0 = x^2 + x^2 h_0(y, dy). \tag{19}$$

To see this directly observe that the Hodge star operator has a similar property

$$\star_g = \star_{g_0} + xA \tag{20}$$

where A is a smooth homomorphism of ${}^{\mathcal{C}}\Lambda^*$.

Exercise 9. See if you can do this reasonably neatly!

This in turn implies that

$$\delta_g = \delta_{g_0} + B,\ B \in \mathrm{Diff}^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*). \tag{21}$$

Thus B has no $1/x$ factor. Now,

$$D \subset x^{\frac{n}{2}} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*) \xrightarrow{B} L^2_g(X; {}^{\mathcal{C}}\Lambda^*) \tag{22}$$

from which it follows that D, D_A and D_R for the metric g are the same as they are for a product metric g_0 with the same limiting metric h_0. Thus we are reduced to the case of a product-type metric for which we were able to do computations using the Mellin transform.

All the terms in the expansion of D, apart from the first, correspond to the poles we discovered in examining the condition $du = \delta_0 u = 0$. We are now working with weaker regularity, namely that du, $\delta_0 u \in L^2_g$. Thus, writing out $u \in D$ in terms of its normal and tangential parts as before tangential parts

$$\begin{gathered}\begin{pmatrix}-d_t & x\partial_x + k\\ 0 & d_t\end{pmatrix}\begin{pmatrix}u_n\\ u_t\end{pmatrix} \in x^{-\frac{n}{2}+1}L^2_{\mathrm{b}}([0,\infty);L^2(\partial X))\\ \begin{pmatrix}-\delta_t & 0\\ -x\partial_x-(n-k) & \delta_t\end{pmatrix}\begin{pmatrix}u_n\\ u_t\end{pmatrix} \in x^{-\frac{n}{2}+1}L^2_{\mathrm{b}}([0,\infty);L^2(\partial X))\end{gathered} \tag{23}$$

The analysis of the (truncated) Mellin transform proceeds very much as before except that the right side in (23) leads only to a holomorphic Mellin transform in $\operatorname{Im} s < -\frac{n}{2}+1$ with L^2 integral on real lines in this set uniformly bounded. Moreover, the invertibility of the full matrix

$$\begin{pmatrix}-d_t-\delta_t & x\partial_x+k\\ -x\partial_x-(n-k) & d_t+\delta_t\end{pmatrix} \tag{24}$$

off the imaginary axis follows as before and it only has a finite number of poles in $-\frac{n}{2} < \operatorname{Im} s < -\frac{n}{2}+1$ of finite multiplicity. Thus we conclude that u_M is meromorpic as a function in $\operatorname{Im} s < -\frac{n}{2}+1$ with values in $H^1(\partial X; {}^{\mathcal{C}}\Lambda^*)$ and su_M is square-integrable, with values in L^2, on real lines except possibly near $\operatorname{Re} s = 0$.

Writing out the steps in the argument we find

(1) From the tangential part of the first condition and the normal part of the second, the coexact part of u_t and the exact part of u_n, in terms of the Hodge decomposition with respect to h_0, must be the Mellin transforms of functions in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(\partial X; {}^{\mathcal{C}}\Lambda^*)$.
(2) The harmonic parts must be such that $(-is+k)u^H_{n,M}$ and $(is-n+k)u^H_{t,M}$ are the Mellin transforms of functions in $x^{-\frac{n}{2}+1}([0,\infty))$ with values in this vector space.
(3) For the exact part of $u_{n,M}$ and the coexact part of $u_{t,M}$ the projection onto the span of the eigenforms with eigenvalues larger than some R are necessarily in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}$. Each of the components corresponding to an eigenvalue λ satisfy the same equation as before with an error in $x^{\frac{n}{2}+1}H^1([0,\infty))$.

So the poles in $\operatorname{Im} s < -\frac{n}{2}+1$ of the Mellin transform of $u \in D$ are therefore precisely the same as thos of the solutions of $du = \delta_0 u = 0$ as analysed before. The terms in the spaces G^*_A, $G^*_R$and E^* have exactly these poles, with arbitrary coefficients of the appropriate type. Thus, subtracting them we may arrange that u_M has no poles below $\operatorname{Im} s = -\frac{n}{2}+1$. However the result may still not be the Mellin transform of a function in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$. However, a similar argument for the poles lying on $\operatorname{Im} s = -\frac{n}{2}+1$ gives rise to terms in $G^*_{\mathcal{L}}$ and $E^*_{\mathcal{L}}$. After subtracting these terms the result is a form in $x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ which shows that D is indeed given by (15).

To shows that D_R is as indicated, we need to show that all terms apart from E_A are contained within it, and that $E_A \cap D_R = \{0\}$. The first requires the construction of approximations $u_j \in \dot{\mathcal{C}}^\infty(X;\Lambda^*)$ such that $u_j \to u$and $du_j \to du$ in L^2_g. For terms in E^*_R a simple cut-off suffices. For terms in $E^*_{\mathcal{L}}$ and $G^*_{\mathcal{L}}$ approximability follows from Lemma 1. For the terms in G^*, which are of the form

$$\chi x^{-is_\lambda}(x^k(-is_\lambda + k)e_\lambda + x^{k-1}dx \wedge e'_\lambda)$$

we first approximate the normal term using a simple cut-off setting

$$u_{j,n} = x^{-is_\lambda}(1 - \chi(x/\epsilon))(-is_\lambda + k)dx \wedge e'_\lambda$$

and then fix the tangential part by solving

$$u_{j,t} = (-is_\lambda + k)\chi(x)x^{-k}\int_0^x t^{k-1-is_\lambda}(1 - \chi(t/\epsilon)dte_\lambda. \tag{25}$$

That □

10. Lecture X; in part

So, I had some difficulties in this lecture! Zhang Zhou pointed out subsequently that the proof of self-adjointness of $d+\delta$ with domain D_R is inadequate. This is a case of me trying to avoid work earlier, only to cause trouble later. The difficulty is that from the definition of D_R^*,

$$D_R^* = \big\{u \in L^2_g(X;\Lambda^*); D_R \ni \phi \longmapsto \langle (d+\delta)\phi, u\rangle \text{ extends by continuity to } L^2_g(X;\Lambda^*)\big\} \tag{1}$$

it does not follow directly that the domain is order-graded. That is, we certainly deduce that $(d+\delta)u \in L^2_g$ for the distributional action of the differential operator but we do not know that du, δu are separately in L^2_g. So, we are forced to go back to the earlier analysis and work harder to derive not just the structure of D but the structure of the, in general larger, space

$$D_{\max}(d+\delta) = \big\{u \in L^2_g(X;\Lambda^*); (d+\delta)u \in L^2_g(X;\Lambda^*)\big\}\,. \tag{2}$$

As I said, I should have done this directly but maybe it is better to postpone it to this point where we have the experience to do it relatively easily.

Proposition 2. *The maximal domain $D_{\max}(d+\delta) = D + U'$ where U' is the finite dimensional vector space which for even n is*

$$U' = \chi(x)\,\mathrm{sp}\{x^{-is_\sigma+k}((is-(n-k-2))\psi_k - dx\wedge d\omega_k)\tag{3}$$

where the linear span is over k and coexact k-forms ψ_k which are eigenforms of the boundary Laplacian, $\Delta\psi_k = \sigma\psi_k$ with

$$-is = -\frac{n}{2} + 1 - \sqrt{(\frac{n}{2}-1-k)^2+\sigma} > -\frac{n}{2}.\tag{4}$$

It follows immediately that such forms can occur only in for dimensions $k = \frac{n}{2}-1$ if n is even of $k = \frac{n}{2} - 1 \pm \frac{1}{2}$ if n is odd. The forms in (3) can never be degree-graded, unless zero of course.

Proof. Elliptic regularity applies as before to show that if $u \in D_{\max}(d+\delta)$ for a conic metric then $u \in x^{-\frac{n}{2}} H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$. Thus, exactly as with the discussion of D, the space $D_{\max}$ is the same for any two metrics with the same boundary metric h_0. We can therefore work with a product-type metric and analyse the conditions under which

$$v = \chi\sum_k (x^k u_{t,k}(x) + x^{k-1}dx\wedge u_{n,k-1}(x))\tag{5}$$

is such that $(d+\delta)u \in L^2_g$, given that u itself is in L^2_g, which is just the condition

$$u_{n,*}, u_{t,*} \in x^{-\frac{n}{2}} L^2_{\mathrm{b}}(X;\Lambda^*(\partial X)).\tag{6}$$

The Hodge decomposition on the boundary allows these tangential and normal parts to be divided. Namely we set

$$L^2(\partial X;\Lambda^*) + L^2(\partial X;\Lambda^*) = H + G + U\tag{7}$$

where H is the harmonic part in all degrees, G is the part we discussed extensively before

$$u \in G \Longleftrightarrow u_n \in dH^1(\partial X;\Lambda^*),\ u_t \in \delta H^1(\partial X;\Lambda^*)\tag{8}$$

and U is the remaining part

$$u \in U \Longleftrightarrow u_n \in \delta H^1(\partial X; \Lambda^*),\ u_t \in dH^1(\partial X; \Lambda^*). \tag{9}$$

The harmonic part is finite dimensional, given by a smoothing operator applied to (u_t, u_n) whereas the components in G and U are given by the action of pseudodifferential projections of order 0. This means that, as necessary, we can track their regularity in Sobolev spaces.

The point of the decomposition (7) is that it is directly related to the action of $d+\delta$, in the product-conic case. Thus, $d_t+\delta_t$ maps exact to coexact forms and conversely so the components of (u_n, u_t) in H, G and U are mapped, respectively, into H, U and G so must separately take values in L^2_g. The H and G components were analysed earlier, so consider the component in U. From the form of d and δ (and taking care to get the value of k right for the action in a given form degree) we arrive at the condition

$$-\delta_0 u_{n,k+1} + (x\partial_x + k)u_{t,k} \in xL^2_g(-x\partial_x - (n-k-2))u_{n,k+1} + du_{t,k} \in xL^2_g \tag{10}$$

as conditions between the tangential component in degree k and the normal component in degree $k+1$ (as opposed to $k-1$ for G.)

As before we analyse the degree to which the condition (10) does *not* imply that $u \in x^{-\frac{n}{2}+1}H^1_{\mathrm{b}}(X; {}^{\mathcal{C}}\Lambda^*)$ by using the Mellin transform to find any possible poles in the strip $-\frac{n}{2} < \operatorname{Im} s \le -\frac{n}{2}+1$. Such poles must satisfy

$$-\delta\phi_{k+1} + (-is+k)\psi_k = 0,\ (is - (n-k-2))\phi_{k+1} + d\psi_k = 0 \tag{11}$$

where ϕ_{k+1} is exact and ψ_k is coexact. Eliminating between the equations as before gives

$$\delta\delta\psi_k + (is - (n-k-2))(-is+k)\psi_k = 0 \tag{12}$$

which is to say that $(is-n+k+2)(is-k) = \sigma$ must be a positive eigenvalue of Δ acting on coexact k-forms. Completing the square we find

$$-is = -\frac{n}{2} + 1 \pm \sqrt{(\frac{n}{2} - 1 - k)^2 + \sigma}. \tag{13}$$

This of course is pure imaginary and can lie in the 'critical strip' only when the sign is $-$ and then only when $k = \frac{n}{2} - \frac{3}{2}$, $k = \frac{n}{2} - 1$ or $k - \frac{n}{2} - \frac{1}{2}$ and only for correspondingly small eigenvalues, namely $\sigma_{\frac{n}{2}\frac{3}{2}} < \frac{3}{4}$, $\sigma_{\frac{n}{2}1} < 1$ and $\sigma_{\frac{n}{2}\frac{1}{2}} < \frac{3}{4}$. In particular there are never such 'accidental poles' on the line $-is = -\frac{n}{2}+1$. These poles can be removed by subtracting a term as in (3). □

Now, the defect form Q is defined on the whole of $D_{\max}$:

$$Q(u,v) = \int_X (\langle (d+\delta)u, v\rangle - \langle u, (d+\delta)v\rangle)\, dg. \tag{14}$$

Moreover by the approximability conditions already discussed it vanishes if either factor is in

$$\begin{aligned} D_{\min}(d+\delta) = \{&u \in L^2_g(X;\Lambda^*);\\ &\exists u_n \to u \text{ in } L^2_g,\ u_n \in \dot{\mathcal{C}}^\infty(X;\lambda^*),\ (d+\delta)u_n \to (d+\delta)u \text{ in } L^2_g\}. \end{aligned} \tag{15}$$

For the moment we know at least that $D_{\min}$ contains all put the finite dimensional parts E^*_A, E^*_R, G^* in D and U^* in $D_{\max}$. There remains a little computation to do:

Lemma 2. *The defect (or boundary) pairing Q defines non-degenerate pairings between E^*_A and E^*_R and also between G^* and U^* and vanishes on all other pairings.*

Proof. Well, we already know the first part. The second part follows by a similar integration-by-parts argument and the eigendecomposition of boundary forms. □

Exercise 10. Carry through the argument here!

So, with this extra work we can see why

$$D^*_R = D_R. \tag{16}$$

Namely, $u \in D^*_R$ certainly implies that $u \in D_{\max}(d+\delta)$. Then the definition implies that Q must vanish on $D_R \times D^*_R$. The fact that $G^* \subset D_R$ and the lemma above then shows that $U^* \cap D^*_R = \{0\}$, which is to say $D^*_R \subset D$ where the previous argument, just the pairing argument for E^*_* takes over and shows that $D^*_R \subset D_R$, and hence they are equal.

Exercise 11. Use the same argument to decide on the exact identity of $D_{\min}(d+\delta)$. Show that an self-adjoint operator $\eth_B$ which is given by $d+\delta$ acting on some domain D_B with $D_{\min} \subset B \subset D_{\max}$ corresponds to a maximal subspace of $E^* + G^* + U^*$ on which Q vanishes.

Since my handwritten lecture notes for today are at best misleading on blow-up of a boundary face of a manifold with corners I have typed up something closer to what I actually said.

Rather than just define $X^2_{\mathrm{b}} = [X^2; (\partial X)^2]$, which we need for the definition of the algebra $\Psi^{-\infty}_{\mathrm{b}}(X)$, I will give the general definition of $[Z, F]$ where F is a boundary face of a compact manifold with corners, Z. By definition (and this is what makes this easier than the general case of an appropriately embedded submanifold) there are global defining functions for F. Namely, if H_i are the boundary hypersurfaces of Z which contain F and x_i are defining functions for the H_i, $i = 1, \dots, k$ then the simultaneous vanishing of the x_i defines F, at least locally near F (there may be other components of the intersection of these k hypersurfaces).

Now to define a manifold it suffices to give the space of smooth functions on it. We can set

$$\begin{multline} \mathcal{C}^\infty([Z; F]) = \\ \{u \in \mathcal{C}^\infty(Z \setminus F); u \text{ is smooth in any normal polar coordinates at a point of } F\}. \end{multline} \tag{17}$$

Here, by normal polar coordinates, I mean polar coordinates in the defining functions x_i. Thus the local coordinates are x_i, y_j, $j = 1, \dots, n-k$. By polar coordinates I will, for the moment, mean 'projective' polar coordinates. These are the k functions

$$r = x_1 + \dots + x_k,\ t_i = \frac{x_i}{r},\ i = 1, \dots, k-1,\ y_j. \tag{18}$$

Notice that (as always locally near F) $r = 0$ only at F. It is only because it is a 'corner' of Z that we can do this. We can replace any one of the 'angular' variables t_i by $t_k = \frac{x_k}{r}$, or more generally take any $k-1$ of these k variables as coordinates. To see that the definition (17) really makes sense, we need to show that it does not actually depend on the choices of the x_i and y_j, although the latter is pretty obvious.

Lemma 3. *If $F : U, 0 \longrightarrow U', 0$ is a diffeomorphism of (relatively) open subsets of $\mathbb{R}^{n,k}$ with $F(0) = 0$ then the pull-back under F of any $\mathcal{C}^\infty$ function of the polar coordinates r, $t_1, \ldots, t_{k-1}$, y_j is also a $\mathcal{C}^\infty$ function of these variables.*

Proof. It suffices to show that the pull-pack functions F^*r, F^*t_i and F^*y_j are $\mathcal{C}^\infty$ functions of r, t_i and y_j, since then the same is true for any smooth function of these variables. It is clear than any relabelling of the x_i has this property, so we can assume that F maps each of the boundary hypersurfaces x_i into itself (rather than permuting them). Thus $F^*x_i = a_i x_i$ with $0 < a_i$ a $\mathcal{C}^\infty$ functions near 0 on $\mathbb{R}^{n,k}$. Since $x_i = t_i r$, it follows that

$$F^*r = \sum_{i=1}^{k} F^*x_i = (\sum_{i=1}^{k} a_i t_i) r = \alpha r,\ 0 < \alpha. \tag{19}$$

Here we use the fact that the t_i, $1 = 1, \ldots, k-1$ take values in the standard simplex in $\mathbb{R}^{k-1}$, i.e. $0 \le t_i \le 1$ and $t_1 + \cdots + t_{k-1} \le 1$. Since $t_k = 1 - t_1 - \cdots - t_{k-1}$ and the a_i in (19) are smooth and positive, it is indeed the case that the coefficient α is positive and a smooth function of the polar variables. From this the rest follows easily, since for instance

$$F^*t_i = \frac{F^*x_i}{F^*r} = \alpha^{-1} a_i t_i. \tag{20}$$

□

Exercise 12. Check that these projective coordinates are equivalent to polar coordinates in the usual sense. That is, show that the functions $R = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ and $\omega_i = x_i/R$ are smooth functions of r and the t_i and conversely. Notice that the ω_i are the coordinates of a vector in $\mathbb{S}^{k-1,k-1} = \mathbb{S}^{k-1} \cap \mathbb{R}^{k,k}$, and that any $k-1$ of the ω_i can be used as coordinates at a point on the sphere, except if there is one which takes the value 1 at the point, in which case only the others form a coordinate system.

Having shown that the definition (17) does actually make sense independent of coordinates, we need to check that the space of functions so defined is indeed the space of all $\mathcal{C}^\infty$ functions on a compact manifold with corners. To make this space concrete we can use the chosen defining functions x_i to identify it as

$$[Z, F] = (Z \setminus F) \cup \mathit{\Delta} \times F,\ \mathit{\Delta} = \{t \in \mathbb{R}^{k-1,k-1}; t_1 + \cdots + t_{k-1} \le 1\}. \tag{21}$$

Proposition 3. *Once the defining functions x_i are fixed, the space in* (17) *gives* $[Z; F]$ *in* (21)*, for F a boundary face of a compact manifold with corners Z, a natural structure as a compact manifold with corners.*

Proof. Already proved really. We can identify a neighbourhood of F in Z with the product $F \times U$ where U is a neighbourhood of 0 in $\mathbb{R}^{k,k}$ of the form $x_1 + \cdots + x_{<}\epsilon$, $\epsilon > 0$. Then the functions r and $t_i = x_i/r$ allows us to identify the part of the union in (21) consisting of $\backslash F$ and $\mathit{\Delta} \times F$ with $F \times \mathit{\Delta} \times [0, \epsilon)_r$. This is consistent with the definition of $\mathcal{C}^\infty([Z; F])$ and so gives the space a $\mathcal{C}^\infty$ structure. □

Exercise 13. If you want to define $[Z; F]$ as a set, canonically and not as in (21) by reference to some particular collection of defining functions, it is not hard to do; so do it! The usual way is to introduce the normal bundle to F. This is the quotient of the tangent bundle to Z over F, $T_F Z$, by the tangent bundle to F. Thus $N_p F = T_p Z / T_p F$ for all $p \in F$. It is a bundle of rank k over F and has a positive 'quadrant' bundle, namely the image of the tangent vectors which satisfy $Vx_i \geq 0$ for all i. This condition is independent of the choice of defining functions x_i. If we let N^+F denote this 'quadrant bundle' we can pass to the corresponding 'fractional sphere bundle' – really a bundle of simplices – given by the quotient by the fibre $\mathbb{R}^+$ action, $SN^+F = (N^+F \setminus 0)/\mathbb{R}^+$. After all this we can set, as a set,

$$[Z;F] = (Z \setminus F) \cup SN^+F. \tag{22}$$

Check that the choice of defining functions x_i gives a natural identification $SN^+F = F \times \Delta$ and the $\mathcal{C}^\infty$ structure induced on $[X;F]$ in (22) by this choice is actually independent of the choice.

Now, the blown up space comes with a smooth map back to the original

$$\beta : [X;F] \longrightarrow Z,\ \beta(r,t,y) = (rt,y), \tag{23}$$

which is independent of any choices, since it is just the canonical identification on the first part of (21), or (22), and the projection onto F on the second part. Under this map, the 'new' boundary face $r = 0$ is identified with F; sometimes I call $r = 0$ the 'front face' of the blow up, or if a sudden algebraic wave overcomes me, the (exceptional) divisor. The lifts (proper transforms) of the old boundary faces $x_i = 0$ are the $t_i = 0$, which are mapped smoothly onto them; I will often use the notation $\beta^*(H)$ for the lift of a boundary hypersurface and $\mathrm{ff}(\beta)$ for the front face. Notice however that

$$\beta^{-1}(\{x_i = 0\}) = \{r = 0\} \cup \{t_i = 0\} = \mathrm{ff}(\beta) \cup \beta^*\{x_i = 0\}, \tag{24}$$

so the preimage of a boundary hypersurface containing F is the union of its lift (proper transform) and the front face (divisor).

Exercise 14. Make sure you see that in this real setting, blowing up a boundary hypersurface does absolutely nothing.

Now, back to the matter at hand. We want do identify the space of 'order $-\infty$ b-pseudodifferential operators' on X with a space of smooth kernels on $X^2_{\mathrm{b}} = [X^2, (\partial X)^2]$. To do so, we should be careful and include the obligator right density factors. Since we are in this 'b-category' it is natural (and wise) to take the density to be a b-density.

To do so, let me introduct another little bit of notation. Since we will need to talk about the projections of X^2 onto the factors, set

$$\pi_R : X^2 \longrightarrow X,\ \pi_R(x,x') = x',\ \pi_L : X^2 \longrightarrow X,\ \pi_L(x,x') = x. \tag{25}$$

Then we need the corresponding 'stretched' maps from X^2_{b} :

$$\pi_{\mathrm{b},R} : X^2_{\mathrm{b}} \longrightarrow X,\ \pi_{\mathrm{b},R} = \pi_R \circ \beta,\ \pi_{\mathrm{b},L} : X^2_{\mathrm{b}} \longrightarrow X,\ \pi_{\mathrm{b},L} = \pi_L \circ \beta. \tag{26}$$

Definition 1. On any compact manifold with boundary we set

$$\Psi^{-\infty}_{\mathrm{b}}(X) = \{A \in \mathcal{C}^\infty(X^2_{\mathrm{b}}; \pi^*_{\mathrm{b},R}\Omega_{\mathrm{b}}); A \equiv 0 \text{ at } \beta^*(\partial X \times X) \cup \beta^*(X \times \partial X)\}. \tag{27}$$

recall that $u \equiv 0$ at H for a smooth function u indicates that it vanishes with its Taylor series at each point of H, so all derivatives vanish there.

Now, as we shall see, these are operators and form an algebra.

Let me show first that the act on the smallest reasonable space, $\dot{\mathcal{C}}^\infty(X)$. It is easy to see that the Schwartz kernel theorem applies here and shows that $A \in \Psi^{-\infty}_{\mathrm{b}}(X)$ does define an operator from $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^{-\infty}(X)$, the space of extendible distributions (the dual of $\dot{\mathcal{C}}^\infty(X;\Omega)$. This is pretty unimpressive, and would not allow us to compose the operators. To do so, observe how the action should go. We want to 'define'

$$Af = \int_X A(z,z')f(z') \tag{28}$$

where $f \in \dot{\mathcal{C}}^\infty(X)$ and I have formally written z, z' for the two 'variables' in X. Notice that the kernel A is supposed to carry within it the density factor needed to carry out the integral.

Trying to interpret (28) rigourously, we have to think of A as a smooth function on X^2_{b}. The functin f is on X but in (28) this is clearly supposed to be interpreted as the right factor of X^2. So we can consider the product

$$A \cdot \pi^*_{\mathrm{b},R} f \in \dot{\mathcal{C}}^\infty(X^2_{\mathrm{b}}; \pi^*_{\mathrm{b},R}\Omega_{\mathrm{b}}). \tag{29}$$

Here we are using the obvious fact that $\pi^*_{\mathrm{b},R}f \in \mathcal{C}^\infty(X^2_{\mathrm{b}})$ which is just the smoothness of the map, but also the fact that it vanishes to infinite order at $\mathrm{ff} \cap \beta^*(X \times \partial X) = \pi^{-1}_{\mathrm{b},R}(\partial X)$, simply because f, by assumption, vanishes to all orders at the boundary. Now, A already vanishes to infinite order at the old boundaries so the product in (29) does, as claimed, vanish to infinite order at all boundaries.

Now, one elementary property of the blow-up procedure is that it induces an isomorphism on functions that are smooth and vanish to infinite order at all boundaries

$$\beta^* : \dot{\mathcal{C}}^\infty(Z) \longleftrightarrow \dot{\mathcal{C}}^\infty([Z;F]) \tag{30}$$

for the blow-up of any boundary face. This allows us to interpret the product in (29) as a section of the b-density bundle

$$Af \in \dot{\mathcal{C}}^\infty(X^2; \pi^*_R\Omega_{\mathrm{b}}) = \dot{\mathcal{C}}^\infty(X^2; \pi_R\Omega) \tag{31}$$

where we use the fact that the sections of Ω_{b} which vanish to infinite order at the boundary are the same, naturally, as the sections of Ω, the ordinary density bundle. Finally then we see that

$$\Psi^{-\infty}_{\mathrm{b}}(X) \times \dot{\mathcal{C}}^\infty(X) \ni (A,f) \longmapsto \int_X Af \in \dot{\mathcal{C}}^\infty(X) \tag{32}$$

is actually a continuous bilinear map. In particular we get the desired operator interpretation

$$A : \dot{\mathcal{C}}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty(X),\ A \in \Psi^{-\infty}_{\mathrm{b}}(X). \tag{33}$$

Exercise 15. Check that this action is faithful, i.e. if A vanishes as an operator (33) then it vanishes as an element of the space $\Psi^{-\infty}_{\mathrm{b}}(X)$.

11. Lecture XI: October 14

Last time I defined (again) the space $\Psi_{\mathrm{b}}^{-\infty}(X)$ and showed that its elements are operators on $\dot{\mathcal{C}}^\infty(X)$. Today I want to prove that they form an algebra and discuss some of its properties, relating them in general to geometric properties of X_{b}^2 ('the b-double space'). As a warm-up exercise, that turns out to be close to the proof of the composition theorem, let me discuss

Proposition 4. *The elements of* $\Psi_{\mathrm{b}}^{-\infty}(X)$ *act on* $\mathcal{C}^\infty(X)$.

Of course this is also important in its own right, as a further justification that the elements of $\Psi_{\mathrm{b}}^{-\infty}(X)$ act on 'almost everything'.

Proof. We are supposed to get this action in the same way as the action on $\dot{\mathcal{C}}^\infty(X)$. Notice that $\pi_{\mathrm{b},R}^* : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X_{\mathrm{b}}^2)$ so the big difference is that we do not have vanishing at the preimage of the boundary, hence not at $\mathrm{ff}(X_{\mathrm{b}}^2)$. We can summarize the operations in the little diagram

(1)
$$\begin{tikzcd} & X_{\mathrm{b}}^2 \arrow[dl, "\pi_{L,\mathrm{b}}"'] \arrow[dr, "\pi_{R,\mathrm{b}}"] & \\ X & & X \end{tikzcd}$$

In fact it is clear from the definition that

$$\Psi_{\mathrm{b}}^{-\infty}(X) \text{ is a } \mathcal{C}^\infty(X_{\mathrm{b}}^2)\text{-module.} \tag{2}$$

Thus, in trying to show that when we integrate $A\pi_{\mathrm{b},R}^* f$ over the right factor of X we get an element of $\mathcal{C}^\infty(X)$, we might as well forget about f and just integrate a general A. Thus we are trying to show that the push-forward map to the left factor gives

$$(\pi_{\mathrm{b},L})_* : \Psi_{\mathrm{b}}^{-\infty}(X) \longrightarrow \mathcal{C}^\infty(X). \tag{3}$$

Note that if the map in question was a fibration, as the left projection from X^2 is, then this is a version of Fubini's theorem. However $\pi_{\mathrm{b},L}$ is *NOT* (quite) a fibration. If it were a fibration then (3) would be true without the vanishing conditions on the 'old boundary faces' which are inherent in the definition of $\Psi_{\mathrm{b}}^{-\infty}(X)$ and $\mathcal{C}^\infty(X_{\mathrm{b}}^2; \pi_{\mathrm{b},R}^*\Omega_{\mathrm{b}})$ itself would push-forward into $\mathcal{C}^\infty(X)$; it does not, so we have to make use of this vanishing.

So, to business. To prove (3) we can work locally on X_{b}^2. Indeed using a partition of unity we can cut the kernel, A, up into small pieces and assume that it has support in the preimage of the product of coordinate neighbourhoods in the two factors of X. If we are away from the front face of X_{b}^2, so away from the corner 'downstairs' in X^2, then (3) is obvious – the map is locally a fibration and in any case we are back to the previous result and the image is actually in $\dot{\mathcal{C}}^\infty(X)$.

Thus, we can assume that A has its support in a 'polar coordinate' neighbourhood $[0,\epsilon)_r \times [-1,1]_t \times U \times U'$ where U, U' are open neighbourhoods of $0 \in \mathbb{R}^{n-1}$ with coordinates y, y' and $r = x+x'$, $t = (x-x')/r$ are projective polar coordinates. Then

$$A = a\frac{dx'}{x'}dy',\ a \in (1-t)^k(1+t)^{k'}\mathcal{C}_c^\infty([0,\epsilon) \times [-1,1] \times U \times U')\ \forall\ k, k'. \tag{4}$$

Here the factors of $1-t$ and $1+t$ reflect the assumed rapid vanishing at the old boundaries, which are $t=\pm 1$. In fact, because we want to discuss the map back to x, y variables, it is convenient to introduce the *singular* projective coordinates

$$s = x'/x, x, y, y' \text{ so } t = \frac{1-s}{1+s},\ r = (s+1)x. \tag{5}$$

These are valid coordinates in some region $0 \le r \le \epsilon$, $t \in (-1, 1]$ with $t = 1, -1$ corresponding to $s = 0, \infty$. My claim is that, despite the singularity of these coordinates, we can translate the conditions on a to imply

$$a'(s, x, y, y') = a(r, t, y, y') \Longrightarrow a' \in \mathcal{S}([0,\infty); \mathcal{C}^\infty_c([0,\delta) \times U \times U')). \tag{6}$$

By this I just mean that a' is $\mathcal{C}^\infty$ has support contained in $[0,\infty) \times K$ for some compact $K \subset [0,\delta) \times U \times U'$ and all derivatives (meaning in s, x, y and y' of all orders) vanish rapidly as $s \to \infty$.

This is clear in $0 \le s \le S$ where the coordinates are legitimate. In $s \ge S > 0$ for any fixed S, we can introduce $s' = 1/s$ $(= x/x')$ taking values in $(0, 1/S)$. We still do not quite get legitimate coordinates since $t = \frac{s'-1}{s'+1}$ is fine, but $r = (1+s')x/s'$ is not smooth. Since $x = x's'$, s', x' *are* legitimate coordinates in this region, with $r = (1+s')x'$ so we do get a smooth function, b, of s', x', y, y' which vanishes to infinite order at $s' = 0$ and has bounded support in x'. Notice that such a function can indeed be written as a smooth function of s', x, y, y' :

$$b'(s', x, y, y') = b(s', \frac{x}{s'}, y, y') \tag{7}$$

because the singularities in the second variable, as $s' \to 0$ are swamped by the rapid decay in s'. For instance we can write

$$b = (s')^N b_N(s', \frac{x}{s'}, y, y'),\ b_N\ \mathcal{C}^\infty,$$

from which it follows that the first $N-1$ derivatives in x are continuous down to $s = 0$. Now, for a function to be smooth and vanish to infinite order at $s' = 0$ is equivalent to its being 'Schwartz' in the variable $s = 1/s'$ near $s = \infty$. Thus we do really have (6).

Exercise 16. Prove the converse to (6) that this (with the correct support constraints) does actually characterize the kernels of elements in $\Psi^{-\infty}_{\mathrm{b}}(X)$.

Finally then we can write our push-forward integral as

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} a(\frac{x'}{x}, x, y, y') \frac{dx'}{x'} dy' \tag{8}$$

where the supports in x' and y' are actually bounded. Changing variable from x' to $s = x'/x$ this becomes

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} a(s, x, y, y') \frac{ds}{s} dy' \tag{9}$$

Note that the measure has 'miraculously' become regular except at $s = 0, \infty$ where we have corresponding rapid vanishing (or decay) in the integrand. Thus the integral (9) converges absolutely and uniformly to a smooth function of x and y. This is what we need to prove. □

This proof is a bit hands-on for my taste! For later purposes I will generalize this result and make it more geometric. The results I will formulate next will first (as usual you might say) be used to prove something worthwhile, in this case the composition theorem, and then later it will be proved. The proof can be based on computations in singular coordinates just like that above, but there are other approaches too.

First think about the properties of smooth maps between compact manifolds with corners. We know what smoothness means already, but we need to add some conditions as to how boundaries are mapped. Recall that the boundary hypersurfaces each have defining functions (if you like these are simply generators of the $\mathcal{C}^\infty$-module of functions which vanish on the boundary hypersurface in question), ρ_H for each $H \in M_1(X)$.

Definition 2. A smooth map $F : X \longrightarrow X'$ is a *b-map* if each boundary defining function $\rho'_{H'}$, $H' \in M_1(X')$ pulls back to a product of boundary defining functions for X :

$$f^*\rho'_{H'} = a_{H'} \prod_{H\in M_1(X)} \rho_H^{e(H,H')},\ a < a_{H'} \in \mathcal{C}^\infty(X). \tag{10}$$

It is an interior b-map if it maps the interior of X into the interior of X'. It is a *simple* b-map if it is a b-map and in addition the exponents $e(H, H')$ take only the values $0, 1$. It is a *b-normal* map if it is a b-map and in addition for each $H \in M_1(X)$ there is at most one $H' \in M_1(X')$ such that $e(H, H') \neq 0$.

A simple b-normal map is one which is simple and b-normal, etc, duh.

Exercise 17. Translate these definitions into statements about the behaviour of the ideals corresponding to boundary faces.

Now recall that the b-cotangent bundle ${}^{\mathrm{b}}T^*X$ is the ordinary cotangent bundle in the interior, but near a boundary face has as local basis the 'logarithmic differentials' dx_i/x_i and dy_j in terms of our usual adapted coordiantes. The b-tangent bundle, its (pre-)dual, has corresponding basis $x_i\partial_{x_i}$, ∂_{y_j}.

Proposition 5. *Any interior b-map the usual differential on the interior extends by continuity to a 'b-differential' and its dual*

$$f^{*\,\mathrm{b}} : {}^{\mathrm{b}}T^*_{f(p)}X' \longrightarrow {}^{\mathrm{b}}T^*_pX,\ f_{*\,\mathrm{b}} : {}^{\mathrm{b}}T_pX' \longrightarrow {}^{\mathrm{b}}T_{f(p)}X, forallp \in X. \tag{11}$$

Note that despite some danger of confusion, I will generally denote this 'new' differential by f^* or f_*, just like the usual one.

Exercise 18. See if you can carry the proof through.

Definition 3. An interior b-map $f : X \longmapsto X'$ is said to be a *b-submersion* if it is surjective and $f_{*\,\mathrm{b}} = f_* : T_pX \longrightarrow T_{f(p)}X'$ is surjective for each $p \in X$. A b-submersion which is also b-normal is said to be a *b-fibration*

Exercise 19. Check that these definitions are not at all vacuous!

(1) Show that the blow-down map $\beta : [X, F] \longrightarrow X$ for F a boundary face of a manifold with corners is always a b-submersion but not a submersion in the usual sense unless F is a boundary hypersurface (in which case it is the identity map).

(2) Show that this blow-down map is never b-normal, and hence is not a b-fibration, unless H is a boundary hypersurface.
(3) Show that the 'stretched projection' $\pi_{L,\mathrm{b}} : X^2_{\mathrm{b}} \longrightarrow X$ is a b-fibration but is not a fibration in the usual sense.

I will discuss the general structure of b-fibrations, and so on, later. For the moment I will just quote a push-forward result

Theorem 1. *For a simple b-fibration f, suppose for each $H' \in M_1(X')$ for which $e(H,H') \neq 0$ for some $H \in M_1(X)$ a particular such $H = p_f(H')$ is chosen, then push-forward (fibre-integration) gives a map*

$$f_* : \big\{u \in \mathcal{C}^\infty(X;\Omega_{\mathrm{b}}); u \equiv 0 \text{ at } H \in M_1(X) \\ \text{unless } H = p_f(H') \text{ for some } H' \in M_1(X')\big\} \longrightarrow \mathcal{C}^\infty(X';\Omega_{\mathrm{b}}). \tag{12}$$

Of course you are very welcome to try to prove this, but it is easier when we have a little more machinery at our disposal. For the moment I suggest

Exercise 20. Show that this theorem does imply Proposition 4 in the form (3). Hint: Since the theorem deals with b-densities and (3) is about 'partial' b-densities, something has to be done! First show that there is a natural isomorphism

$$(\pi_{L,\mathrm{b}})^*\Omega_{\mathrm{b}}(X) \otimes (\pi_{R,\mathrm{b}})^*\Omega_{\mathrm{b}}(X) \equiv \Omega_{\mathrm{b}}(X^2_{\mathrm{b}}) \tag{13}$$

(Hint-within-a-hint, the corresponding statement on X^2 is true). Now to get (3), choose a positive b-density $0 < \nu_{\mathrm{b}} \in \mathcal{C}^\infty(X;\Omega_{\mathrm{b}})$ and show that Theorem 1 can be applied to $\Psi^{-\infty}_{\mathrm{b}}(X) \cdot \pi^*_{L,\mathrm{b}}\nu_{\mathrm{b}}$. Check that the result is independent of the choice of ν_{b}.

Despite appearances there is something going on here to do with b-densities as opposed to ordinary densities.

12. Lecture XII, 16th October, 2003

Handwritten notes: Pages 1-14

13. Lecture XIII, 21 October, 2003

Handwritten notes: Pages 1-12

14. Lecture XIV, 23 October, 2003

What have I not done to complete the treatment of the 'conic case' at least as far as the identification of the L^2 cohomology, the relative Hodge cohomology and the appropriate intersection cohomology is concerned? I have not

(1) Treated the composition of finite-order b-pseudodifferential operators.
(2) This in turn is only really needed (for the moment) for the proof of the Sobolev continuity of such operators, that $A \in \Psi_{\mathrm{b}}^k(X)$ always defines a bounded linear operator from $x^s H_{\mathrm{b}}^m(X)$ to $x^s H_{\mathrm{b}}^{m-k}(X)$ for any m, s. This is what gives us elliptic regularity.
(3) I have not yet discussed intersection cohomology at all.

I will add to the notes, but probably not devote a lecture to, the first two of these. My reasoning here is that these reduce, given what we already know, to the same issues in the boundaryless case, so I do not feel the need to go through the discussion fully here.

Rather than go through the conic case again, I will now quickly describe the same sort of approach to another class of degenerate metrics which I will call 'cusps' but are often called 'horns'. I do not want to take the time to go through all the details, but I will attempt to write down everything to the point where it is 'straightforward' to check the claims that I make.

The metrics we consider again exist on any compact manifold with boundary, but with a somewhat different degeneration than for conic metrics. Thus, we suppose that in the interior, g is a metric and near the boundary there is a boundary defining function ρ such that

$$g = d\rho^2 + \rho^{2N} h \tag{1}$$

where h is as before, a smooth symmetric 2-cotensor which restricts to the boundary to a metric h_0 and $N \geq 2$; the conic case corresponds to $N = 1$. The extreme case, $N = 0$, is that of a regular boundary problem, which can also be handled the same way but leads to somewhat different analytic issues (and a different L^2 cohomology of course, namely the absolute cohomology).

Exercise 21. Let $Y^n \subset X^{n+1}$ be a singular submanifold of a compact manifold without boundary where Y has just an isolated singular point (or perhaps several) near which there are local coordinates z_j, in which it takes the form

$$z_0^{2N} = \sum_{j=1}^{n} z_j^2 + f(z_1, \ldots, z_n),\ z_0 \geq 0, \tag{2}$$

where f (real-valued) vanishes to order 3 at least at 0. Show that the introduction of the singular coordinates z_0, z_j/z_0^N resolves Y to a manifold with boundary to which a metric on X restricts to a 'horn' metric (note that z_0 might not quite be x.)

For extra credit (!) show that the same thing can be accomplished by repeatedly blowing up the singular point, namely it needs to be blown up N times.

Problem 1. Describe the L^2 and Hodge cohomology for a metric of this 'cusp' type. In fact we want to do 'everything' in a sense that should be getting clearer by now.

The approach I will use is, and of course this is one of the main points, essentially the same as in the conic case although some of the 'details' are necessarily different.

Namely, first look at the structure of $d+\delta$ in terms of a Lie algebra of vector fields such that we can give an elliptic regularity result for the associated enveloping algebra (and develop a full calculus of pseudodifferential operators to go along with this). This is used to analyse the relative and absolute domains, which have the same definitions as before, deduce self-adjointness and the Fredholm property and hence get the Hodge decomposition and identity of L^2 cohomology and relative Hodge cohomology. It turns out that in this case the Hodge cohomologies can be identified in terms of the usual relative/absolute cohomology and subsequently in terms of appropriate intersection cohomology. Rather surprisingly perhaps the spaces (say L^2 cohomology) on a fixed manifold for different metrics and different values of $N \geq 1$ turn out to be canonically isomorphic.

So, first we look for a Lie algebra of smooth vector fields with which to describe the Laplacian and $d+\delta$. If we look at vector fields of finite length they will generally be singular at the boundary, with the worst singularity being $O(\rho^{-N})$ (take $N=2$ if you want). So we can look at the vector fields V which are smooth and satisfy

$$|V|_g = O(\rho^N). \tag{3}$$

Since the part $\rho^{2N}h$ of the metric already gives such an order of vanishing for any smooth V, this is equivalent to

$$V\rho \in \rho^N \mathcal{C}^\infty(X) \Longleftrightarrow V \in \mathcal{V}_{\mathrm{Nc}}(X). \tag{4}$$

Locally, in adapted coordinates, in which $x=\rho$ must always be an *admissible* defining function, i.e. one for which (4) holds, this Lie algebra and $\mathcal{C}^\infty(X)$ module is spanned by

$$x^N\partial_x,\ \partial_{y_j}. \tag{5}$$

It follows that it is the space of all smooth sections of a vector bundle, ${}^{\mathrm{Nc}}TX$, for which (5) gives a local basis. In the case $N=2$ I introduced this Lie algebra long ago; it depends on the choice of ρ as a trivialization of the normal bundle to the boundary, but nothing more. For $N \geq 3$ it only depends on the choice of ρ modulo terms $O(\rho^N)$ as is clear from (4).

Exercise 22. Can you give a 'geometric' description of an N-cusp structure on a compact manifold with boundary, analogous to the trivialization of the normal bundle in case $N=2$?

Similarly we define the N-cusp cotangent, and form, bundles based on $\mathcal{C}^\infty$ combinations of the forms

$$dx,\ \rho^N dy_j \tag{6}$$

I will denote these bundles ${}^{\mathrm{Nc}}T^*X$ and ${}^{\mathrm{Nc}}\Lambda X$; note that they depend on more than N!

Since ${}^{\mathrm{Nc}}T^*X$ is, by definition, the dual of ${}^{\mathrm{Nc}}TX$, a smooth section of the latter, $V \in \mathcal{V}_{\mathrm{Nc}}(X)$, defines a smooth function on the former which is linear on the fibres; we normalize this by defining $\sigma(V)$ to be iV thought of as a linear function. A function $f \in \mathcal{C}^\infty(X)$ similarly defines a smooth function on ${}^{\mathrm{Nc}}T^*X$ which is constant on the fibres (and we do not put an i in the identification of $\sigma(f)$ with f in this sense). Let $\mathrm{Diff}^k_{\mathrm{Nc}}(X)$ be the space of N-cusp differential operators of order k (at most). Thus $P \in \mathrm{Diff}^k_{\mathrm{Nc}}(X)$ is an operator, for example on $\mathcal{C}^\infty(X)$, which can be written as a finite sum of up to k fold products of elements of $\mathcal{V}_{\mathrm{Nc}}(X)$; this one can think of as the enveloping algebra of $\mathcal{V}_{\mathrm{Nc}}(X)$ as a Lie algebra and $\mathcal{C}^\infty(X)$ module (in particular $k = 0$ factors means the action of $f \in \mathcal{C}^\infty(X)$ by multiplication); the $\mathrm{Diff}^k_{\mathrm{Nc}}(X)$ clearly form a (n order-)filtered algebra. Moreover, from the fact that $\mathcal{V}_{\mathrm{Nc}}(X)$ is a Lie algebra

$$[\mathrm{Diff}^k_{\mathrm{Nc}}(X), \mathrm{Diff}^l_{\mathrm{Nc}}(X)] \subset \mathrm{Diff}^{k+l-1}_{\mathrm{Nc}}(X) \tag{7}$$

we see that the commutative product in $\mathcal{C}^\infty({}^{\mathrm{Nc}}T^*X)$ leads to a short exact sequence

$$\mathrm{Diff}^{k-1}_{\mathrm{Nc}}(X) \hookrightarrow \mathrm{Diff}^k_{\mathrm{Nc}}(X) \xrightarrow{\sigma_k} \mathcal{P}^k({}^{\mathrm{Nc}}T^*X). \tag{8}$$

Here the quotient space is the space of smooth functions on ${}^{\mathrm{Nc}}T^*X$ which are homogeneous (polynomials) of degree k on the fibres. The symbol map is *determined* by the fact that it is multiplicative and our earlier normalization on $\mathcal{C}^\infty(X) = \mathrm{Diff}^0_{\mathrm{Nc}}(X)$ and $\mathcal{V}_{\mathrm{Nc}}(X)$.

We can extend these definitions to sections of vector bundles without pain. Either localize everything, which is a bit painful, or interpret the tensor product in

$$\mathrm{Diff}^k_{\mathrm{Nc}}(X; E, F) = \mathrm{Diff}^k_{\mathrm{Nc}}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X; \hom(E, F)). \tag{9}$$

Of course it is important that this defines a space of operators $\mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F)$.

Exercise 23. Check that there are no surprises and the symbol extends in the obvious way and gives rise to a short exact sequence as in (8) but with bundles inserted appropriately.

As usual, ellipticity means precisely that $\sigma_k(P)$ is invertible off the zero section of ${}^{\mathrm{Nc}}T^*X$.

Now we look at $d + \delta$ from this point of view.

Lemma 4. *For an N-cusp metric* (1), $d + \delta \in \rho^{-N} \mathrm{Diff}^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*)$ *is elliptic in this sense.*

Proof. To check that $d \in \rho^{-N} \mathrm{Diff}^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*)$ just work out its action on the local basis of 1-forms (6):

$$\begin{aligned} d(adx + \sum_{k=1}^{n-1} b_k x^N dy_k) = x^{-N}(&\sum_{j=1}^{n-1} (\partial_{y_j} a) x^N dy_j \wedge dx \\ &+ \sum_{k=1}^{n-1} (x^N \partial_x b_k + N x^{N-1} b_k) dx \wedge x^N dy_k + \sum_{l,k=1}^{n-1} (\partial_{y_l} b_k) x^N dy_l \wedge x^N dy_k \end{aligned} \tag{10}$$

Remark 2. It is precisely at this point that we see a simplification arising in the cases $N \geq 2$ relative to the conic case, $N = 1$. Namely in the middle, 'cross', term here the term of order 0, which arises from the x-differentiation of x^N in the basis of forms, vanishes at $x = 0$ if $N > 1$. This means that this term will not show up in the 'model' operator we later consider, as we considered the model cone earlier. For this reason the non-zero eigenvalue problems that appeared for the cone, and caused most of the computation work, do not show up at all for $N > 1$. I did say the cone was the hardest case earlier! It also means that the 'model operator' for $d + \delta$, when we try to look at what happens to leading order at the boundary is not, or at least should not be thought of, as the operator for the model problem.

For the latter this x^{N-1} term would appear, but it is irrelevant to the analysis and is better dropped. Make of that what you will.

Then we can either check, exactly as before, that Hodge $\star$ is an isomorphism of ${}^{\mathrm{Nc}}\Lambda^* X$, which is essentially immediate from the definition, or else that the adjoints with respect to a measure such as

$$dg = \rho^{N(n-1)}\nu,\ 0 < \nu \in \mathcal{C}^\infty(X;\Omega), \tag{11}$$

and non-degenerate fibres inner products, of elements of $\mathrm{Diff}^k_{\mathrm{Nc}}(X;E,F)$ are in $\mathrm{Diff}^k_{\mathrm{Nc}}(X;F,E)$. Anyway, we easily conclude that $\delta \in \mathrm{Diff}^1_{\mathrm{Nc}}(X;{}^{\mathrm{Nc}}\Lambda^*)$.

The argument for ellipticity is the same as before. We can see from (10) that the symbol of d at (p,ξ), $\xi \in {}^{\mathrm{Nc}}T^*_pX$, is $i\xi\wedge$ acting on ${}^{\mathrm{Nc}}\Lambda^*_pX$. The symbol of the adjoint is the adjoint of the symbol (or use $\star$) so the symbol of δ is $-ic_\xi$ in terms of metric contraction. The result is a Clifford action of ${}^{\mathrm{Nc}}T^*X$ on ${}^{\mathrm{Nc}}\Lambda^*X$, and in any case is elliptic since its square is diagonal and given by multiplication by the metric (remember, this is a course on Dirac operators, except I have only talked about one so far!) □

We want uniform elliptic estimates (and more, we really want a way to write down the inverse of $d+\delta$). To get these we will work on a double space which resolves $\mathcal{V}_{\mathrm{Nc}}(X)$. This is supposed to be obtained through iterated blow-up, $\beta : X^2_{\mathrm{Nc}} \longrightarrow X^2$ and be such that we can lift $\mathcal{V}_{\mathrm{Nc}}$ smoothly from either factor of X and the resulting smooth vector fields are transversal to the lifted diagonal. We also want the stretched projections $\pi_{H,\mathrm{Nc}} = \pi_H \circ \beta$, $H = L, R$, to be b-fibrations which are transversal to the lifted diagonal (among other things this means that the lifted diagonal is diffeomorphic to X). Let's try to do it; maybe just sticking to $N = 2$ would be wise, but I will go on and outline the general case.

Locally our vector fields are $x^N\partial_x$ and ∂_{y_j}. Just as in the conic case we do not want, in fact cannot, do anything to the tangential ∂_{y_j} vector fields, since they are already non-degenerate. Basically this resolution problem is again 1-dimensional, or since we are in the double space, 2-dimensional, just involving x and x'. An obvious thing to look at is the lift of $x^N\partial_x$ to the space X^2_{b}, which resolves $\mathcal{V}_{\mathrm{b}}(X)$. In the coordinates $s = x/x'$ x', y_j and y'_j near a point on the lifted diagonal $s = 1$, $y = y'$ in X^2_{b} we know that $x\partial_x = s\partial_s$. (So that the lift of $\mathcal{V}_{\mathrm{b}}(X)$ is everwhere transversal to the diagonal). So of course, $x^N\partial_x$ lifts to $(x')^{N-1}s^N\partial_s$. Since $s = 1$ on the diagonal, this vanishes exactly at $x' = 0$ on the lifted diagonal, which is to say at its boundary. However, we cannot blow-up the boundary of the diagonal, since the ∂_{y_j} are not tangent to it! The smallest reasonable thing to blow up is

$$B_2 = \{x' = 0, s = 1\} \subset \mathrm{ff}(X^2_{\mathrm{b}}). \tag{12}$$

Exercise 24. Check that this is actually a well-defined submanifold of X^2_{b} which depends (only) on the choice of cusp structure, i.e. the defining function ρ. Note that it is a boundary p-submanifold, i.e. is an interior p-submanifold of the boundary hypersurface $\mathrm{ff}(X^2_{\mathrm{b}}$.

Even though the notation is not quite defined, we consider

$$X^2_{\mathrm{cu}} = [X^2_{\mathrm{b}}; B_2]. \tag{13}$$

I have not defined the blow up of a boundary p-submanifold such as B but it is a straightforward generalization of the blow up of a boundary face. We get a new boundary face ff and b-map as blow-down map. The main (big) difference is that this map is not a b-submersion, if fact it is a b-submersion exactly when B is a boundary face, which it is not here. This is replaced by the fact that

Lemma 5. *The Lie algebra $\mathcal{V}_{\mathrm{b}}(X;Y)$ of vector fields tangent to the boundary faces of X and to the p-submanifold Y lifts smoothly to $[X;Y]$ to span $\mathcal{V}_{\mathrm{b}}([X;Y])$ as a module over $\mathcal{C}^\infty([X;Y])$.*

Thus we do in fact know the range of β_*.

Exercise 25. Check this in the particular case of interest here, namely for $B_2 \subset X^2_{\mathrm{b}}$.

Proposition 6. *The cusp algebra $\mathcal{V}_{\mathrm{cu}}(X)$ defined by a choice of boundary defining function $\rho \in \mathcal{C}^\infty(X)$ on a compact manifold with boundary lifts, from the right or left factor, to a space of smooth vector fields on X^2_{cu} (defined of course by the same choice of ρ) to be transversal to the lifted diagonal, which is an interior p-submanifold* $\mathrm{Diag}_{\mathrm{cu}} \subset X^2_{\mathrm{cu}}$. *The left and right stretched projections, $\pi_{O,\mathrm{cu}} = \pi_O \circ \beta_{\mathrm{cu}}$, $O = L, R$, are b-fibrations which are transversal to* $\mathrm{Diag}_{\mathrm{cu}}$.

Proof. That the cusp algebra lifts, we know from Lemma 5. In any case it is a rather straightforward computation which I will do! We can ignore ∂_{y_j} throughout and we have only to deal with the one vector field, which starts off as $x^2\partial_x$. After we lift it to X^2_{b} it is $x's^2\partial_s$ in terms of the coordinates $x', s = x/x'$ which are valid near the boundary of the diagonal. We can drop the s^2 since it is non-vanishing and switch from s to $t = s-1$ which has the virtue of vanishing at $B_2 = \{x' = 0, t = 0\}$ and our vector field is a non-vanishing smooth multiple of $x'\partial_t$. The lifted diagonal is $y = y'$ $t = 0$ and near it we can use the singular coordinates x' and $t_2 = t/x'$ (together with y and y'). In terms of these our vector field has become ∂_{t_2}, so with ∂_{y_j} we do indeed get a set of smooth vector fields transversal to the interior p-submanifold $\mathrm{Diag}_{\mathrm{cu}}$.

So, X^2_{cu} does resolve $\mathcal{V}_{\mathrm{cu}}(X)$. Now we need to check that we haven't gone too far somehow. So, $\pi_{R,\mathrm{cu}}$ is a well-defined b-map. Why is it a b-submersion? Consider the vector field $x\partial_x + x'\partial_{x'}$. This is in $\mathcal{V}_{\mathrm{b}}(X^2)$ and so lifts to X^2_{b} to be smooth. Near B_2 it has become

$$(t-1)\partial_t + (x'\partial_{x'} - (t-1)\partial_t) = x'\partial_{x'}.$$

in terms of the coordinates $t = s-1 = (x-x')/x'$ and x'. Here the first term is the lift of $x\partial_x$ and the second is the lift of $x'\partial'_x = -s\partial_s + x'\partial_{x'}$ in the new coordinates. Thus it is certainly tangent to B_2 and so lifts to be smooth on X^2_{cu} by Lemma 5. But this means that the vector field to which it lifts pushes forward under $\pi_{L,\mathrm{cu}}$ (or $\pi_{R,\mathrm{cu}}$ for that matter) to $x\partial_x$ on X. So in fact $(\pi_{L,\mathrm{cu}})_* : {}^{\mathrm{b}}T_pX^2_{\mathrm{cu}} \longrightarrow {}^{\mathrm{b}}T_{p'}X$ must always be surjective! Since the image manifold is a manifold with boundary the additional condition of b-normality is void. Thus $\pi_{L,\mathrm{cu}}$ is a b-fibration. That this b-fibration is transversal to $\mathrm{Diag}_{\mathrm{cu}}$ is the statement that the null space of the (ordinary or b-) differential $(\pi_{R,\mathrm{cu}})_*$ contains a complement to the tangent space of $\mathrm{Diag}_{\mathrm{cu}}$ at each point. This we already know, since the lifts from the left factor of elements of $\mathcal{V}_{\mathrm{cu}}(X)$ must be killed by $\pi_{R,\mathrm{cu}}$, and this lift spans such a complement at each point. □

Now, having done this in the cusp case, $N = 2$, I may as well go on into the higher cusp cases. First try $N = 3$. Then we have $x^3\partial_x$ in place of $x^2\partial_x$. So, when we lift it up to X^2_{cu} from the left, we can see from the computation above that in the coordinates t_2, x' (and always y, y') that we get a smooth positive multiple of $x'\partial_{t_2}$. So, we have to blow up $B_3 = \{t_2 = 0, x' = 0\} \subset X^2_{\mathrm{cu}}$. Thus we conclude that

$$\mathcal{V}_{3\mathrm{c}}(X) \text{ is resolved on } X^2_{3\mathrm{c}} = [X^2_{\mathrm{cu}}; B_3]. \tag{14}$$

In fact the same argument clearly works for any N. Proceeding by induction we can claim that the functions

$$t_N = \frac{x - x'}{(x')^N}, x', y, y' \tag{15}$$

lift to X^2_{Nc} to give coordinates near the lifted diagonal in which it becomes $t_N = 0$, $y = y'$ and such that $x^{N+1}\partial_x$ lifts from the left factor to be a smooth positive multiple of $x'\partial_{t_N}$. Then we can define $B_{N+1} = \{x' = 0, t_N = 0\}$ and define the next space as

$$X^2_{(\mathrm{N}+1)\mathrm{c}} = [X^2_{\mathrm{Nc}}; B_N]. \tag{16}$$

Proposition 7. *Proposition 6 carries over to the N-cusp algebra with X^2_{cu} replaced by X^2_{Nc}.*

With this behind us, we can define

$$\Psi^k_{\mathrm{Nc}}(X) = \{A \in I^k(X^2_{\mathrm{Nc}}, \mathrm{Diag}_{\mathrm{Nc}}; \pi^*_{R,\mathrm{Nc}}\Omega_{\mathrm{Nc}}); A \equiv 0 \text{ at } \partial X^2_{\mathrm{Nc}} \setminus \mathrm{ff}_{\mathrm{Nc}}\}, \tag{17}$$

where $\Omega_{\mathrm{Nc}} = \rho^{-Nn+1}\Omega_{\mathrm{b}} = \rho^{-Nn}\Omega$.

Exercise 26. I leave it to you to show how to define the operators on sections of vector bundles.

There are lots of things to say about these operators, and I will say at least some of them. The first thing is to see that $\mathrm{Diff}^k_{\mathrm{Nc}}(X) \subset \Psi^k_{\mathrm{Nc}}(X)$. The place to start here is the identity operator! In local coordinates it can be written

$$\mathrm{Id}\, u(x, y) = \int \delta(x - x')\delta(y - y')u(x', y')|dx'dy'|. \tag{18}$$

To lift the kernel up to X^2_{b} we need to introduce say $s = x/x'$ as variable in place of x. Since it is essentially a parameter we can use the fact that the delta 'function' is homogeneous of degree -1, so

$$\delta(x - x') = (x')^{-1}\delta(s - 1) = (x')^{-1}\delta(t). \tag{19}$$

But the factor of x' just turns dx' into dx'/x' and we get a coefficient b-density:

$$\delta(t)\delta(y - y')|\frac{dx'}{x'}dy'| \in \Psi^0_{\mathrm{b}}(X). \tag{20}$$

We can continue this way up to X^2_{Nc} to see that

$$\mathrm{Id} = \delta(t_n)\delta(y - y')|\frac{dx'}{(x')^N}dy'| \in \Psi^0_{\mathrm{Nc}}(X). \tag{21}$$

Clearly it is elliptic, since it has symbol 1.

Exercise 27. Now use the fact that $\mathcal{V}_{\mathrm{Nc}}(X)$ lifts from the left fact to be smooth vector fields inn $\mathcal{V}_{\mathrm{b}}(X^2_{\mathrm{Nc}})$ to show that $\mathrm{Diff}^k_{\mathrm{Nc}}(X) \subset \Psi^k_{\mathrm{Nc}}(X)$.

Proposition 8. *The elements of $\Psi^m_{\mathrm{Nc}}(X)$, for any $m \in \mathbb{R}$, define continuous linear operators on $\mathcal{C}^\infty(X)$.*

Proof. This is an application of the push-forward theorem using the fact that the stretched projections are b-fibrations; it is also necessary to sort out the behaviour of the density factors. □

Proposition 9. *The $\Psi^k_{\mathrm{Nc}}(X)$ form an order-filtered asymptotically complete *-algebra of operators on $\mathcal{C}^\infty(X)$ with multiplicative symbol map giving a short exact sequence*

$$\Psi^{m-1}_{\mathrm{Nc}}(X) \hookrightarrow \Psi^m_{\mathrm{Nc}}(X) \longrightarrow (S^m/S^{m-1})({}^{\mathrm{Nc}}T^*X) \tag{22}$$

and each $A \in \Psi^0_{\mathrm{Nc}}(X)$ is bounded on $L^2(X)$.

Proof. So, I have left a bit of a hole in the preparation for the product formula – in particular I don't quite have the machinery in place to prove even the composition formula in the boundaryless case, so I will have to talk about that too. So this whole proof will take a little while – maybe you should bypass it as I will do in the lecture!

First we consider the composition formula for operators of order $-\infty$, which of course is part of the claim. The idea here is exactly the same as before. We want to find a triple product with appropriate properties. This will involve a bit of an effort. Let's start with the triple b-product X^3_{b}. We already know that X^3_{b} maps back under $\pi O, \mathrm{b}$ to X^2_{b}. Now, inside X^2_{b} we have the submanifold B_2 that we need to blow up to turn X^2_{b} into X^2_{cu}. So, we consider the inverse image $\pi^{-1}_{O,\mathrm{b}}(B_2) = B_{2,O}$ for $O = F, S, C$. Two of the boundary faces of X^3_{b} are mapped into the front face of X^2_{b} under each of the stretched projections so we actually get two, intersecting, boundary p-submanifolds as the preimage of B_2 from each of the projections. Somewhere there are some pictures of what is going on!

□

In particular our elliptic construction from the boundaryless case carries over unchanged.

Proposition 10. *If $P \in \Psi^k_{\mathrm{Nc}}(X; E, F)$ is elliptic then there exists $Q \in \Psi^{-k}_{\mathrm{Nc}}(X; F, E)$ such that $P \circ Q - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{Nc}}(X; F)$ and $Q \circ P - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{Nc}}(X; E)$.*

One important point is that, as in the conic case, $(x/x')^s$ is a multiplier on the space $\Psi^k_{\mathrm{Nc}}(X; E, F)$, although as we shall see, much more is true for $N > 1$. Anyway, by conjugating and using ellipticity it follows that

$$u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*),\ \rho^N(d+\delta)u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*) \Longrightarrow u \in \rho^{-\frac{Nn}{2}} H^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*) \tag{23}$$

Here I have put in the weight that comes from the metric. The Sobolev space on the right is just that based on $\mathcal{V}_{\mathrm{Nc}}(X)$, so $u \in H^p_{\mathrm{Nc}}(X)$ just means that $\mathcal{V}_{\mathrm{Nc}}(X)^j u \subset L^2_{\mathrm{Nc}}(X)$ for all $j \le p$. This of course applies to the maximal, ungraded, domain

$$D_{\max} = \left\{u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*); (d+\delta)u \in L^2_g(X; {}^{\mathrm{Nc}}\Lambda^*)\right\} \subset \rho^{-\frac{Nn}{2}} H^1_{\mathrm{Nc}}(X; {}^{\mathrm{Nc}}\Lambda^*) \tag{24}$$

So, we then want to work out more precisely what this domain, and the various smaller ones we have defined, D, D_A, D_R and $D_{\min}$ are. This turns out to be fairly straightforward when $N \ge 2$ using the finer conjugation property anticipated above. Namely, for any real τ,

$$\exp(i\tau(\frac{1}{x} - \frac{1}{x'}) \text{ is a multiplier on } \Psi^k_{\mathrm{cu}}(X). \tag{25}$$

More generally

$$\exp(i\tau(x^{-N}-x'^{-N}) \text{ is a multiplier on } \Psi^k_{\mathrm{Nc}}(X). \tag{26}$$

Exercise 28. Check this by seeing what happens to the function when lifted to X^2_{Nc}; i.e. show that it is smooth except at the part of the boundary where the kernels are assumed to vanish rapidly where it only has a singularity of finite order.

Now, the maximal graded domain

$$D = \left\{u \in L^2_g(X;\Lambda^*); du, \delta u \in L^2_g(X;\Lambda^*)\right\} \tag{27}$$

with norm $\|u\|^2_D = \|u\|^2_{L^2} + \|du\|_{L^2} + \|\delta u\|_{L^2}$ and the relative and absolute domains

(28)
$$D_R = \{u \in D; \exists \dot{\mathcal{C}}^\infty(X;\Lambda^*) \ni u_n \to u \text{ in } L^2_g(X;\Lambda^*) \text{ s.t. } du_n \to du \text{ in } L^2_g(X;\Lambda^*)\},\ D_A = *D_R.$$

Lemma 6. *If $k = \frac{n-1}{2}$ (so n is odd) and $U : H^k_{\mathrm{Ho},h_0}(\partial X;\Lambda^k) \longrightarrow D$, and $V : H^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) \longrightarrow D$ such that*

$$U\phi = \phi + \rho\mathcal{C}^\infty(X;\Lambda^k),\ V\phi = d\rho\wedge\phi + \rho\mathcal{C}^\infty(X;\Lambda^{k+1}). \tag{29}$$

Proof. For $\phi \in H^k_{\mathrm{Ho},h_0}(X)$, let $\phi(x)$ be the representative harmonic with respect to the varying metric $h(x,y,dy,0)$. The termss in dx can be suppressed since these are already $O(x^2)$ with respect to the metric. It then follows that $U\phi = \chi\phi(x)$, for an appropriate cut-off χ, is in D and then we can take $V\phi = *U\phi$. □

Basically there is nothing else!

Theorem 2. *For a cusp metric as in* (1) *the graded L^2 domain*

$$D = \overline{x^{-\frac{n}{2}+1}H^1_{\mathrm{cu}}(X;{}^{\mathrm{cu}}\Lambda^*)} + UH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) + VUH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X), \tag{30}$$

where the closure is with respect to $\|\cdot\|_D$, and the relative domain

$$D_R = \overline{x^{-\frac{n}{2}+1}H^1_{\mathrm{cu}}(X;{}^{\mathrm{cu}}\Lambda^*)} + UH^{\frac{n-1}{2}}_{\mathrm{Ho},h_0}(\partial X) \tag{31}$$

and with this domain, $d+\delta$ is a self-adjoint Fredholm operator with consequent Hodge decomposition

$$L^2(X;{}^{\mathrm{cu}}\Lambda^*) = H^*_g(X) \oplus dD_R \oplus \delta D_R \tag{32}$$

and null space canonically isomorphic to the L^2 deRham cohomology.

Proof. This involves computations similar to, but easier than, those in the conic case. □

15. Lecture XV, 28 October, 2003

Handwritten notes: Pages 1-9

16. Lecture XVI, 30 October, 2003

Handwritten notes: Pages 1-9

17. Lecture XVII, 4 November, 2003

Next lecture will be Thursday November 13 (and it will be a short one!)

Last time I constructed a principal bundle associated to any family $A \in \mathcal{C}^\infty(Y; \Psi^k(X; E, F))$ of elliptic pseudodifferential operators on a compact manifold without boundary, X, ofr a compact parameter space Y. The *structure group* of this bundle is $G^{-\infty}(X; E)$ or $G^{-\infty}(X; F)$ as the numerical index of the family if negative or positive. Assuming for the same of definiteness that $\# - \operatorname{ind}(A) \geq 0$, the bundle

$$\begin{tikzcd} G^{-\infty}(X; F) \arrow[r] & \mathcal{P}_{A,N} \arrow[d] \\ & Y \end{tikzcd} \tag{1}$$

has fibre at $y \in Y$

$$\mathcal{P}_{A,N,y} = \left\{ B \in \Psi^{-\infty}(X; E, F); A_y + B \text{ has null space exactly } N \right\} \tag{2}$$

where $N \subset \mathcal{C}^\infty(X; E)$ is fixed but is chosen arbitrarily with dimension equal to $\# - \operatorname{ind}(A.)$.

Essentially by definition, this bundle is trivial if and only if there exists a smooth map $E \in \mathcal{C}^\infty(Y; \Psi^{-\infty}(X; E, F))$ such that $A_y + E_y$ is surjective for all $y \in Y$ and has null space N.

Exercise 29. Check this carefully, starting from the definition of triviality of a principal bundle.

Thus, the triviality of the principal bundle, together with the vanishing of the numerical index is *precisely* the obstruction to 'perturbative invertibility'.

Also recall that I *defined*

$$K^{-1}(Y) = [Y; G^{-\infty}] \tag{3}$$

$$K^{-2}(Y) = [Y \times \mathbb{S}, Y \times \{1\}; G^{-\infty}, \operatorname{Id}] \tag{4}$$

i.e. $K^{-1}(Y)$ is the set of homotopy classes of smooth maps into $G^{-\infty}$ (for any model) and $K^{-2}(Y)$ is similarly the set of homotopy classes of smooth maps from $Y \times \mathbb{S}$ into $G^{-\infty}$ taking $Y \times \{1\}$ to Id. We can also thing of (4) as

$$K^{-2}(Y) = [Y; \mathcal{L}G^{-\infty}, \operatorname{Id}] \tag{5}$$

where $\mathcal{L}G^{-\infty}$ is the loop group:

$$\mathcal{L}G^{-\infty} = \left\{ F : \mathbb{S} \longrightarrow G^{-\infty}; F(1) = \operatorname{Id} \right\}. \tag{6}$$

The definitions (3) and (4) depend on the fact, which is the essential nature of *Bott Periodicity* that

$$\Pi_j(G^{-\infty}) = \begin{cases} \{0\} & j \text{ even} \\ \mathbb{Z} & j \text{ odd.} \end{cases} \tag{7}$$

Exercise 30. Assuming (7) show that

$$\Pi_j(\mathcal{L}G^{-\infty}) = \begin{cases} \mathbb{Z} & j \text{ even} \\ \{0\} & j \text{ odd} \end{cases} \tag{8}$$

where the higher homotopy groups can be considered as maps into the component of the identity.

Exercise 31. What are $K^{-1}(\mathbb{S}^n)$ and $K^{-2}(\mathbb{S}^n)$?

I will prove the first part of (7). The first claim is

Lemma 7. *$G^{-\infty}$ is connected.*

Proof. This is a direct consequence of the fact that for any smoothing operators $A \in \Psi^{-\infty}(X; E)$ the family $\operatorname{Id} + zA$ is invertible for $A \in \mathbb{C} \setminus D$ where $D \subset \mathbb{C}$ is discrete (i.e. countable and without points of accumulation). We can either use the Fredholm determinant to prove this or proceed directly. Fix a value $\bar{z}$ of z. If $\operatorname{Id} + zA$ is invertible then we know it has a bounded inverse as an operator on $L^2(X; E)$ and by the openness of the set of invertible operators (i.e. convergence of the Neumann series) it remains invertible for $|z - \bar{z}| \|A\| \|(\operatorname{Id} - \bar{z}A)^{-1}\| < 1$. Thus D is closed. If $\operatorname{Id} + \bar{z}A$ is not invertible then we use finite rank approximation to write $A = A_1 + A_2$ where A_1 is a finite rank smoothing operator and A_2 has small norm, for instance $|\bar{z}| \|A_2\|_{L^2} < \frac{1}{2}$. Then $\operatorname{Id} + zA_2$ has inverse $\operatorname{Id} + B(z)$ for $|z - \bar{z}| < \epsilon$ with $B(z)$ holomorphic with values in the smoothing operators and we are reduced to considering

$$(\operatorname{Id} + B(z))(\operatorname{Id} + zA) = \operatorname{Id} + A'(z),\ \ A'(z) = (\operatorname{Id} + B(z))zA_1$$

Thus $A'(z)$ has finite rank, at most the rank of A_1, and is holomorphic near z so $\operatorname{Id}_N + A'(z)$ is invertible for $0 < |z - \bar{z}| < \epsilon'$ for some $\epsilon > 0$ and D is discrete.

Thus if $\operatorname{Id} + A$ is invertible then it can be connected to the identity by an inverible family $\operatorname{Id} + z(s)A$. $\square$

Exercise 32. Use such a finite rank approximation to define the Fredholm determinant $\det(\operatorname{Id} + A)$ as an entire function of A, extending the usual definition for finite rank operators, such that $(\operatorname{Id} + A)^{-1}$ exists if and only if $\det(\operatorname{Id} + A) \neq 0$ with the usual multiplicative and differential properties

$$\begin{gathered}\det((\operatorname{Id} + A)(\operatorname{Id} + B)) = \det(\operatorname{Id} + A)\det(\operatorname{Id} + B),\\ \frac{d}{dz}\det(\operatorname{Id} + zA) = \det(\operatorname{Id} + zA)\operatorname{Tr}(\operatorname{Id} + zA)^{-1}A) \text{ where } \det(\operatorname{Id} + zA) \neq 0.\end{gathered} \tag{9}$$

The second, and more substantial, part of (7) is

Proposition 11. *If $F : \mathbb{S} \longrightarrow G^{-\infty}(X; E)$ is a smooth loop then*

$$w(F) = \frac{1}{2\pi i}\int_{\mathbb{S}} \operatorname{Tr}\left(F(\theta)^{-1}\frac{dF(\theta)}{d\theta}\right) d\theta \in \mathbb{Z} \tag{10}$$

there exists a smooth map

$$\tilde{F} : [0, 1] \times \mathbb{S} \longrightarrow G^{-\infty}(X; E) \text{ with } \tilde{F}(0) = F \text{ and } \tilde{F}(1) = (\operatorname{Id} - \pi) + z^{w(F)}\pi \tag{11}$$

where π is a projection of rank one.

We start off with a simple case, where the family is actually affine.

Lemma 8. *If A, B are $N \times N$ complex matrices and $A + zB$ is invertible on $|z| = 1$ then, for $|z| = 1$, it is homotopic to $(\operatorname{Id} - \pi) + z\pi$ where π is a projection of rank $w(A + zB)$.*

Proof. If A is not invertible, we may perturb the family slightly and so deform it to $(A + t\operatorname{Id}) + zB$ where the constant term is invertible. Then, using the connectedness of $\mathrm{GL}(N)$, which follows from the proof above, we may deform away the constant term and replace the family by $\operatorname{Id} + zB'$, $B' = (A + t\operatorname{Id})^{-1}B$. On the circle $z = e^{i\theta}$,

$dz = ie^{i\theta}d\theta$ so for a family which is holomorphic near the circle the integral in (10) can be written as a contour integral

$$w(F) = \frac{1}{2\pi i}\int_{|z|=1} \operatorname{Tr}\left(F(\theta)^{-1}\frac{dF(z)}{dz}\right) dz \tag{12}$$

In this case $dF/dz = B$ and the integral, without the trace, becomes

$$M = \frac{1}{2\pi i}\int_{|z|=1} (\operatorname{Id} + zB')^{-1}B'dz \tag{13}$$

which is in fact the projection onto the span of the generalized eigenspaces of B for outside the unit circle (and with null space the span of those inside). We don't need all of this information but we do need to see that M is a projection (or perhaps better to say an idempotent, $M^2 = M$). Indeed the square can be written as the double integral

$$M^2 = \frac{1}{(2\pi i)^2}\int_{|z|=1}\int_{|z'|=1+\epsilon} (\operatorname{Id} + zB')^{-1}(\operatorname{Id} + z'B')^{-1}(B')^2 dzdz' \tag{14}$$

for $\epsilon > 0$ small (using Cauchy's theorem). Now the resolvent identity can be written

$$(\operatorname{Id} + zB')^{-1}(\operatorname{Id} + z'B')^{-1}B' = (z'-z)^{-1}\left((\operatorname{Id} + zB)^{-1} - (\operatorname{Id} + z'B')\right),\ z \neq z'. \tag{15}$$

Inserting this into (14), one of the integrals can be carried out for each term. Indeed the second is holomorphic in $|z| \le 1$ so integrates to zero, while for the first has a simple pole in z' at $z' = z$ and so the z' integral may be replaced by the residue which is just M.

Furthermore M and B' commute, since B' commutes with $(\operatorname{Id} + zB')^{-1}$ and

$$(\operatorname{Id} + zB')^{-1}M \text{ is holomorphic in } |z| \ge 1,\ (\operatorname{Id} + zB')^{-1}(\operatorname{Id} - M) \text{ is holomorphic in } |z| \le 1. \tag{16}$$

This involves an argument similar to that above, to prove the first write

$$\begin{aligned}(\operatorname{Id} + zB')^{-1}M &= \frac{1}{2\pi i}\int_{|s|=1} (\operatorname{Id} + zB')^{-1}(\operatorname{Id} + sB')^{-1}B'ds \\ &= \frac{1}{2\pi i}\int_{|s|=1} (s-z)^{-1}\left((\operatorname{Id} + zB')^{-1} - (\operatorname{Id} + sB')^{-1}\right) ds.\end{aligned}$$

Here the first term vanishes (for $|z| > 1$), by Cauchy's theorem, and the second is holomorphic. The other case is similar.

Finally, we conclude that under the deformation $B'_t = t(\operatorname{Id} - M)B' + M(tB' + 2(1-t))$ $(\operatorname{Id} + zB'_t)^{-1}$ remains holomorphic near $|z| = 1$ and results in a family as desired. □

18. Lecture XVIII, 13 November, 2003

Handwritten notes: Pages 1-11

19. Lecture XIX, 18 November, 2003

Handwritten notes: Pages 1-10

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