Versions

(1) 30 November, 2021: L1 and compactification
Outline and Practicalities

In this course I hope to cover four (types of) theorems which involve microlocal analysis and in particular the theory of pseudodifferential operators. Explicitly

1. Hörmander’s theorem on the propagation of singularities
2. Weyl’s law for the distribution of eigenvalues
3. The Atiyah-Singer index theorem and K-theory
4. Hodge theory and boundaries

To me these are all fundamental results, and I like them! The first two are relatively closely related and both give realization of the ‘semiclassical limit’, the interplay between the non-commutative theory of (pseudo-)differential operators and the more familiar behaviour of analysis of functions. The latter two are more global.

Clearly, each of these could easily expand to take the whole semester. Still I hope to show how they can be approached using pseudodifferential operators and ‘quantization’. In fact an alternative title for this course might be ‘Smooth quantization’. So most of the time will be devoted to preparing the background material, specifically pseudodifferential operators on $\mathbb{R}^n$, pseudodifferential operators on a manifold, families of pseudodifferential operators and then rings of pseudodifferential operators quantizing a Lie algebroid.

I plan to give 26 one-hour lectures in the 9:30-10:30 slot on Tuesdays and Thursdays and leave 20 minutes for questions and discussions (even short presentations by students); if there is sufficient interest I will organize another ‘discussion’ time, perhaps on Wednesdays in the afternoon. There will be notes for each lecture, which may include topics I will not have time to cover and will certainly include further references – to books, lecture notes and papers. With any luck most of the lectures should appear on my webpage before the beginning of the semester.

Problem sets: There will be approximately 10.

Grades: Graduate students are expected to participate actively. That is what ‘A’ means to me. By this I mean that I expect people to attend lectures and to ask questions. For undergraduates this course might be heavy lifting, it is for me, so please talk to me by early in the semester at the latest. We can discuss what you should expect. There are no exams.

Prerequisites: I will assume familiarity with manifolds and distributions, essentially as in 18.155 but plan to review pretty much everything.

Why don’t I just follow a book or my earlier lecture notes? This probably reflects some personal failing and general disfation with how things are done! I find it difficult to thing through things without seeing some other way of approaching them. If it is not to your taste, I am sorry but that is the way it is. I may not get to all the result listed above, but I expect to at least get to the point where they are all within reach and that is really what I want to do – try to put these results in a general context that maybe encourages them to be exploited (i.e. applie) and extended.

In the interim, feel free to contact me with questions or comments.

CHAPTER 1

Manifolds, distributions and operators

1. L1

The main aim of this course is to describe various algebras of pseudodifferential operators. Let me start with a traditional ‘crypto-historical’ description of the ‘standard’ algebra of pseudodifferential operators on \( \mathbb{R}^n \). I recall notation for functions below, but let’s assume you know about the space of functions

\[
\mathcal{C}_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset \mathcal{C}_c^\infty(\mathbb{R}^n)
\]

maybe including their topologies and duals.

For any multiindex \( \alpha \in \mathbb{N}_0^n \), \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) being the non-negative integers, the corresponding iterated partial derivative acts on each of these space

\[
u \mapsto -\overrightarrow{D}_\alpha u, \quad D_\alpha u(x) = i^{-|\alpha|} \frac{\partial^\alpha_1}{\partial x_1} \cdots \frac{\partial^\alpha_n}{\partial x_n} \cdots u(x), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n
\]

where the normalizing power of \( i \) is inserted to help with notation for the Fourier transform.

Similarly, each of the spaces in (1) is a ring, so multiplication of functions is defined. Combining these we consider \( \text{linear partial differential operators} \) which are given by sums

\[
Pu = \sum_{|\alpha| \leq m} p_\alpha(x) D^\alpha u.
\]

In each case when the coefficients are in one of the spaces (1) we get an operators – a continuous linear map – on the corresponding space.

Now, let’s concentrate on the Schwartz space. For this we have the Fourier transform

\[
F : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n), \quad F u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} dx.
\]

It is a linear isomorphism. We know that

\[
u \in S(\mathbb{R}^n) \implies F(D^\alpha u)(\xi) = \xi^\alpha \hat{u}(\xi).
\]

The Fourier transform conjugates differentiation to multiplication. Of course a monomial such as \( \xi^\alpha \) is not in the Schartz space, but it does define an operator on it by multiplication.

So the inverse Fourier transform allows us to write

\[
D^\alpha u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^\alpha \hat{u}(\xi) d\xi.
\]
Now, the partial differential operator is given by a finite sum so we can combine (6) with (3) and write

\[ Pu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} p(x,\xi) \hat{u}(\xi) d\xi, \quad p(m,\xi) = \sum_{|\alpha| \leq m} p_\alpha(x)\xi^\alpha d\xi. \]

Since \( \hat{u} \in \mathcal{S}(\mathbb{R}^n) \), the integral converges absolutely. We just assume that the coefficients are in \( \mathcal{C}^\infty(\mathbb{R}^n) \) then the integral converges uniformly on compact subsets in \( x \in \mathbb{R}^n \), with all its formal derivatives in \( x \) because of the obvious estimates

\[ |D_x^\gamma p(x,\xi)| \leq C_{K,\gamma}(1 + |\xi|)^m, \quad x \in K \subset \mathbb{R}^n, \quad xi \in \mathbb{R}^n. \]

We can actually define the ‘standard’ space of pseudodifferential operators of order \( m \in \mathbb{R} \) by considering those functions \( a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) which satisfy the symbol estimates

\[ |D_x^\gamma D_\xi^\delta a(x,\xi)| \leq C_{\beta,\gamma}(1 + |\xi|)^{m-|\beta|}, \quad \forall \gamma,\beta \in \mathbb{N}_0^n. \]

Notice that \( p \) in (7) satisfies these estimates for an integer \( m \) if the coefficients are in the space

\[ \mathcal{C}_{\infty}(\mathbb{R}^n) = \{ f \in \mathcal{C}^\infty(\mathbb{R}^n); \sup |D_x^\gamma f(x)| < \infty \forall \gamma \} \]

the space of smooth functions with all derivatives bounded.

The space of functions satisfying estimates (9) is often written \( S^m_{1,0} \) as part of a more general class of spaces \( \mathcal{S}^m_{p,\delta} \) where the exponent \( m - |\beta| \) is replaced by \( m - \rho|\beta| + \delta|\alpha| \). I will probably not talk about these – in fact there are many variants of such estimates (see for instance \( ? \)).

Now, it follows directly that if \( a \in S^m_{1,0} \), in the sense that all the estimates (9) hold, then the direct generalization of (7),

\[ Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi) \hat{u}(\xi) d\xi \Rightarrow a : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{C}_{\infty}(\mathbb{R}^n). \]

In fact more is true

**Theorem 1.** The space of operators, \( \Psi^m_{1,0}(\mathbb{R}^n) \) defined by symbols \( a \) satisfying (9) act on \( \mathcal{S}(\mathbb{R}^n) \) and form a filtered ring

\[ \Psi^m_{1,0}(\mathbb{R}^n) \circ \Psi^{m'}_{1,0}(\mathbb{R}^n) \subset \Psi^{m+m'}_{1,0}(\mathbb{R}^n), \quad \forall \ m, m' \in \mathbb{R}. \]

This is the main content of the first chapter of [7], see also [7]. Probably the first place this result appeared in this form is [7].

I am going to approach this result somewhat indirectly, in the sense that I want to give a good deal of background before proving it. In fact I am more interested in the smaller space which I will denote just \( \Psi^m(\mathbb{R}^n) \) often called the ring of ‘classical’ pseudodifferential operators where the symbols \( a \) have the additional property:

There exists a sequence \( a_i \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \) of homogeneous functions of degree \( m - i \) (in the \( \xi \) variables)

\[ a_i(x, t\xi) = t^{m-i}a(x, \xi), \quad t > 0, \quad (x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \]

such that for (any) cutoff \( \chi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) with \( \chi = 1 \) near 0

\[ a(x, \xi) - \sum_{i=0}^N (1 - \chi(\xi))a_i(x, \xi) \in S^{m-N-1}_{1,0}. \]
These form a filtered subring $\Psi^m(\mathbb{R}^n) \subset \Psi^m_{1,0}(\mathbb{R}^n)$ which for positive integral $m$ includes the differential operators of order $m$ discussed above.

These rings have many important properties but one is that one can recover the terms $a_i$ in (14) from the operator $A$ and the leading term defines the principal symbol as a map

\[
\Psi^m(\mathbb{R}^n) \longrightarrow \{ a_0 \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \text{ homogeneous of degree } m \text{ in } \xi \}
\]

and this map is surjective, multiplicative and defines a short exact sequence

\[
\begin{align*}
\sigma_{m+m'}(A \circ B) &= \sigma_m(A)\sigma_{m'}(B), \quad A \in \Psi^m(\mathbb{R}^n), \ B \in \Psi^{m'}(\mathbb{R}^n) \\
\Psi^{m-1}(\mathbb{R}^n) &\hookrightarrow \Psi^m(\mathbb{R}^n) \longrightarrow \{ a_0 \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \text{ homogeneous of degree } m \text{ in } \xi \}
\end{align*}
\]

Here I have stuck with a cumbersome notation for the homogeneous space which will be refined below.

So, we want to prove all these things and a lot more! However, I do not want to go there directly but rather map out the territory a bit first, in particular discussing the ‘symbol spaces’ concretely.

2. Manifolds with corners

This might appear to be a serious non-sequitor but I hope you will see a bit later why I am proceeding this way. In fact there are several reasons. First to understand the symbol spaces and their generalizations and so to introduce the spaces of conormal distributions which arise as the Schwartz kernels of the operators we are interested in and allow their generalization. This involves manifolds with boundary, but then products will get you to manifolds with corners.

So, this is one of the basic settings for the course – analysis on manifolds with corners – but only taken as far as we need for the moment.

3. Compactification

Although we will deal with non-compact manifolds, the ones that arise below have some ‘structure at infinity’. One way to describe what this means is through the notion of compactification.

**Definition 1.** A compactification of a manifold $M$ is a compact manifold $\overline{M}$ and a smooth injection $\iota : M \rightarrow \overline{M}$ which is a diffeomorphism to a (relatively of course) open dense submanifold.

Here both $M$ and $\overline{M}$ may have corners. As always we need to specify when two compactifications are ‘the same’.

**Definition 2.** Two compactifications $\iota_i : M \rightarrow \overline{M}_i$ are equivalent if there exists a diffeomorphism $e : \overline{M}_1 \rightarrow \overline{M}_2$ giving a commutative diagramme

\[
\begin{array}{ccc}
\overline{M}_1 & \xrightarrow{e} & \overline{M}_2 \\
\downarrow{\iota_1} & & \downarrow{\iota_2} \\
M & \xrightarrow{\iota_2} & \overline{M}_1.
\end{array}
\]
Notice that the equivalence map \( e \) is unique if it exists since it is fixed on an open dense subset by (17). We also say that one compactification is \textit{finer} than another if there is a smooth map \( e \) giving a commutative diagram; again it if it exists it is determined. This defines a partial order on compactification – as we shall see below there can be non-comparable compactifications.

If \( M \) is compact it is a compactification of itself and it is unique in this sense of equivalence.

We might well want more structure for the compactification – for instance if \( M \) is a complex manifold then we might want \( M \) to be complex and all maps to be holomorphic. There are important examples from algebraic geometry here. Most relevant at the moment is the projective compactification of a complex vector space \( W \mapsto \mathbb{P}W \) which I mention below but there are much more sophisticated examples to check out. There is the Deligne-Mumford compactification of the Riemann moduli spaces \( \mathcal{M}_{g,n} \) (okay I here a complaint from someone that the \( \mathcal{M}_{g,n} \) are not quite manifolds, they are orbifolds in general, but take the number of punctures \( n \) large compared to the genus \( g \geq 0 \)). Also there is the deConcini-Procesi ‘wonderful’ compactification of complex adjoint Lie groups \( ... \) if you are interested look also the real version of this in \( ... \)). Also, compactification of ‘Gravitational Instantons’ (aren’t the Physicists good at inventing names!).

The examples I will consider immediately are more prosaic, namely of a real finite-dimensional vector space \( V \). This is both to illustrate the notion and for later reference. I will discuss

(1) The one-point compactification(s) given by a sphere \( \overline{V}^o \).
(2) The parabolic compactification given a closed ball \( \overline{V}^p \).
(3) The radial compactification also given by a closed ball \( \overline{V} = \overline{V}^R \).

From the notation you can see that I have a preference for the radial compactification – I hope the discussion below shows why. Only the radial compactification is really used subsequently.

These can all be constructed using variants of stereographic projection. So, let’s start with \( V = \mathbb{R}^n \), i.e. choose a basis. We embed \( \mathbb{R}^n \) into \( \mathbb{R}^{n+1} \) as the hyperplane

\[ \mathbb{R}^n \ni x \rightarrow (x,1) \in P \subset \mathbb{R}^{n+1}. \]

In the first case consider the the sphere \( S_o \) of radius \( \frac{1}{2} \) centred at \( (0, \frac{1}{2}) \) and in the second and third cases take the sphere \( S_R \) of radius 1 centred at the origin. In both cases a point of \( \mathbb{R}^n \) determines a unique line \( L_o(x) \) or \( L_R(x) \) through the image of \( x \) in \( P \) and the centre of the corresponding sphere then

\[ I_o : \mathbb{R}^n \rightarrow S_o, \quad I_o x \text{ is the other point in } S_o \cap L_o(x) \]

\[ I_R : \mathbb{R}^n \rightarrow S_R^+, \quad I_R x \text{ is the other point in } S_R \cap L_R(x) \subset S_R^+ = S_R \cap \{ x_n+1 \geq 0 \} \]

\[ I_p : \mathbb{R}^n \rightarrow \mathbb{B}_p \subset \mathbb{R}^n, \quad I_p x \text{ is the projection of } L_R x \text{ onto the closed unit ball in } \mathbb{R}^n \times \{ 0 \}. \]

In all three cases the full orthogonal group \( O(n) \), acting on the first factor of \( \mathbb{R}^n \times \mathbb{R} \) satisfies \( I_\bullet Ax = AL_\bullet x \) for all \( A \in O(n) \), effectifely reducing the discussion to the
cae $n = 1$. Explicit formulæ for the maps are easily derived:

$$I_o x = \left( \frac{x}{1 + |x|^2}, \frac{1}{1 + |x|^2} \right) \in S_o$$

$$I_R = \left( \frac{x}{(1 + |x|^2)^{1/2}}, \frac{1}{(1 + |x|^2)^{1/2}} \right) \in S_R^+$$

$$I_p x = \frac{x}{(1 + |x|^2)^{1/2}} \in \mathbb{R}^n.$$

Thus, for the radial compactification $(1 + |x|^2)^{-1/2}$ is a boundary defining function and hence $|x|^{-1}$, which is a smooth function of it away from $x = 0$, is a defining function near the boundary. It follows that

$$\{ |x| > \epsilon > 0 \} \ni x \mapsto \left( \frac{1}{|x|}, \frac{x}{|x|} \right) \in [0, 1) \times S^{n-1}$$

extends to a smooth product decomposition of $\mathbb{R}^n_R$ near the boundary. For the parabolic compactification it follows similarly that

$$\{ |x| > \epsilon > 0 \} \ni x \mapsto \left( \frac{1}{|x|^2}, \frac{x}{|x|} \right) \in [0, 1) \times S^{n-1}$$

is a product decomposition near the boundary.

It can be seen directly that

$$I_o \left( \frac{x}{|x|^2} \right) = S I_o \text{ where } S : S_o \setminus \{(0,1),(0,0)\} \rightarrow S_o \setminus \{(0,1),(0,0)\},$$

with $S(y, y_n) = (y, -y_n + 1)$

is equatorial reflection on $S_o$.

In all cases it is clear either geometrically, or from the formulæ (20), that the action of $O(n)$ extends smoothly from $\mathbb{R}^n$ to the compactification. Similarly the scaling action by $\mathbb{R}^+$, with generator on $\mathbb{R}^n$

$$\sum x_i \frac{\partial}{\partial x_i}$$

extends smoothly. For the one-point compactification this follows from (23) and in the other two cases

$$\lim_{|x| \rightarrow \infty} \frac{tx}{(1 + t|x|^2)^{1/2}} = \frac{x}{(|x|^2)^{1/2}} \text{ and } \lim_{|x| \rightarrow \infty} \frac{1}{(1 + t|x|^2)^{1/2}} = 0.$$

Thus in all cases the action of the conformal group $O(n) \times \mathbb{R}^+$ extends smoothly to the compactification.

**Proposition 1.** The action of the general linear group extends smoothly from $\mathbb{R}^n$ to the radial and parabolic compactifications but not to the one-point compactification; the translation action of $\mathbb{R}^n$ extends smoothly to the radial and the one-point
compactifications but not to the parabolic compactification and there are smooth surjective maps, which are not diffeomorphisms, giving a commutative diagramme (26)

\[
\begin{array}{ccc}
\text{GL}(n,R) \rtimes \mathbb{R}^n & \xrightarrow{} & \\
\downarrow & & \\
\mathbb{S}^n_+ & \xrightarrow{\iota_R} & \mathbb{R}^n \times \mathbb{I} \xrightarrow{\iota_p} \mathbb{B}^n \xrightarrow{} \text{GL}(n,R).
\end{array}
\]

Outline of proof. That the group actions extend as indicated follows by noting that the Lie algebra of \(\text{GL}(n,R)\) consists of vector fields homogenous of degree 0 and similarly the translations are homogeneous of degree \(-1\). Similar arguments show that the groups shown are the maximal subgroups of \(\text{GL}(n,R) \rtimes \mathbb{R}^n\) which extend to act smoothly on the one-point and parabolic compactifications. □

Corollary 1. The one-point compactification is defined for a vector space with conformal-Euclidean structure, the radial compactification is well-defined for an affine space and the parabolic compactification is well-defined for a vector space.

Both the radial and the parabolic compactifications have boundaryless variants, in which the bounding sphere is replaced by an embedded projective space \(\mathbb{S}^{n-1}/\pm\) by doubling across the boundary. The apparent advantage of this smaller compactification does not seem to be realized in practice.

Conjecture 1. The five compactifications are minimal in their respective categories (i.e. as manifolds with/without boundary) among compactifications with the invariance properties in (26).

Although, as noted above, it is the radial compactification which mostly appears below other variants are relevant. In particular none of these compactifications are natural for products – the radial compactification of \(V_1 \times V_2\) is not ‘comparable’ to the products of the radial compactifications. Still this relationship is significant and is examined below.