CHAPTER 3

Cones, horns and cusps

1. Conic metrics

Now that we have examined the 'standard' class of metrics on a compact manifold with boundary, I want to bring out the fact that this is by no means the only class of metrics which one can consider even when, as here, we restrict attention to incomplete metrics.[1] Next I will discuss metrics which make the boundary 'look like a point', whereas the smooth metrics make the boundary look like a submanifold of codimension one.

Consider a point, p, in a Riemann manifold. Riemannian normal coordinates can be introduced around p and then the metric looks flat to first order, i.e.

$$g_{ij} = \delta_{ij} + O(|x|^2)\,.$$

Introducing polar coordinates, $x = r\omega$, $\omega \in \mathbb{S}^{n-1}$, the metric becomes

$$g = dr^2 + r^2\left(d\omega^2 + dr\cdot\gamma + O(r)\right)$$

where $d\omega^2$ is the round metric on the sphere. The introduction of polar coordinates 'blows up' the point so that the manifold now looks like $[0,1)_r \times \mathbb{S}^{n-1}$ nearby. This is the trivial case of a conic metric on a manifold with boundary.

Definition 1. If X is a manifold with boundary a *conic metric* on X is a smooth symmetric 2-cotensor, g, which is positive-definite in the interior and which takes the form

$$g = dx^2 + x^2 h \quad \text{near } \partial X \tag{3.1}$$

in terms of some product decomposition, with h a smooth symmetric 2-cotensor which restricts to $x = 0$ to give a metric on ∂X.

I will discuss the Hodge theory for such a conic metric. In fact it is convenient, as in the non-degenerate case, to consider metrics which are 'product conic' near the boundary in the sense that (3.1) is refined to

$$g = dx^2 + x^2 h_0\,, \quad \text{near } \partial X\,, \tag{3.2}$$

with h_0 a metric on ∂X. Really I want mainly to talk about the *model case* which is of this type but on the manifold

$$g = dx^2 + x^2 h \quad \text{on } c(Y) = [0,\infty)\times Y \tag{3.3}$$

so $Y = \partial c(Y)$. Of course this manifold is not compact.

There is a fundamental difference between the boundary case previously considered and the conic case, in that the latter is intrinsically global in terms of the cross-section. Thus we cannot replace Y by Euclidean space, but rather we need to *use* the compactness of Y.

[1]Thanks to Sang Chin and Andras Vasy for helpful comments on this section.

The model conic metric (3.3) encourages us to use a slightly different 'weighted' decomposition of the form bundles on $[0,\infty)\times Y = c(Y)$; namely we insert factors of x:

$$\begin{gathered}\Lambda^* c(Y) \cong \Lambda^* Y \oplus \Lambda^* Y \\ \Lambda^k c(Y) \ni u = x^k\varphi + \, dx\wedge x^{k-1}\psi,\ \ \varphi\in\Lambda^k Y\,, \psi\in\Lambda^{k-1}Y\,.\end{gathered} \tag{3.4}$$

This has the effect of removing the conformal factors of x from the form inner products

$$\langle u, u'\rangle_g = \langle\varphi,\varphi'\rangle_h + \langle\psi,\psi'\rangle_h\,.$$

The Riemann density of g is

$$dg = x^{n-1}\,dx\,dh$$

where dh is the Riemann density of h.

The decomposition (3.4) reduces d to a 2×2 matrix of operators on x-dependent sections of $\Lambda^k Y\oplus\Lambda^{k-1}Y$; namely

$$\begin{pmatrix} \frac{d}{x} & 0 \\ \partial_x + \frac{k}{x} & -\frac{d}{x}\end{pmatrix}.$$

The factors of $1/x$ here arise from the dependence of the power on the form degree in (3.4); the extra term k/x has the same origin. If u is a k-form and u' is a $(k+1)$-form this shows that, assuming supports are compact

$$\begin{gathered}(du\,,u') \\ = \int_0^\infty\int_y\left(\frac{1}{x}\langle d\varphi\,,\varphi'\rangle + \langle\partial_x\varphi + \frac{k}{x}\varphi\,,\psi'\rangle - \langle\frac{d}{x}\varphi\,,\psi'\rangle\right)x^{n-1}\,dx\,dh\,.\end{gathered}$$

Assuming that u' (or u) vanishes to all orders at the boundary there are no boundary terms on integration by parts so

$$\begin{aligned}(du\,,u') = (u\,,\delta u') = \int_0^\infty\int_Y\Big(&\langle\varphi\,,\frac{\delta}{x}\varphi'\rangle \\ &+\langle\varphi\,,-\partial_x\psi' - (n-1-k)\frac{\psi'}{x}\rangle - \langle\varphi\,,\frac{\delta}{x}\psi'\rangle\Big)\,x^{n-1}\,dx\,dh\,.\end{aligned}$$

Thus, as a 2×2 matrix acting on $\Lambda^{k+1}Y\oplus\Lambda^k Y$ the operator δ becomes

$$\begin{pmatrix} \frac{\delta}{x} & -\partial_x - \frac{(n-1-k)}{x} \\ 0 & -\frac{\delta}{x}\end{pmatrix}.$$

Combining these two formulae shows that the action of $d+\delta$ on $\Lambda^k c(Y)$ is represented under the identification (3.4) with the 2×2 matrix D where

$$xD = \begin{pmatrix} d+\delta & -x\partial_x - (n-K-1) \\ x\partial_x + K & -(d+\delta)\end{pmatrix} \text{ on } \Lambda^k Y\oplus\Lambda^{k-1}Y, \tag{3.5}$$

K being the number operator on forms on the boundary.

We are primarily interested in identifying appropriate boundary conditions for D (and hence of course $d+\delta$). As we shall see this reduces to the analysis of the null space of D, so I shall proceed to describe this.

Observe that D is *almost* a differential operator in x with coefficients depending only on $d+\delta$. I say *almost* because it also depends on the form degree k. So, suppose $V \subset \mathcal{C}^\infty(Y\,;\Lambda^* Y)$ is a subspace which is decomposed under its form gradation:

$$V = \oplus V_k\,,\quad V_k = V \cap \mathcal{C}^\infty(Y;\Lambda^k)$$

and in addition V is invariant under $d+\delta$:

$$(d+\delta)V \subset V\,.$$

Then D acts on $\mathcal{C}^\infty\left([0,\infty)_x; V \oplus V\right)$. In particular if we decompose $\mathcal{C}^\infty(Y;\Lambda)$ into such spaces V then the null space of D on the various $\mathcal{C}^\infty\left([0,\infty)_x; V \oplus V\right)$ will, collectivly, span the null space of D.

The obvious examples of such V, and the most important ones, are the harmonic subspaces. Thus if $\varphi \in H^k_{\mathrm{Ho}}(Y)$ is chosen then D acts on sections of the form

$$f(x)x^k\varphi + g(x)\,dx \wedge x^k\varphi\,, f, g \in \mathcal{C}^\infty\left((0,\infty)\right)$$

Thus[2]

$$D\begin{pmatrix} f(x)\varphi \\ g(x)\varphi \end{pmatrix} = \begin{pmatrix} \left(-\partial_x g - \frac{n-k-1}{x}g\right)\varphi \\ \left(\partial_x f + \frac{kf}{x}\right)\varphi \end{pmatrix},$$

so the null forms are spanned by

$$\begin{pmatrix} x^{-k}\varphi \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ x^{-n+k+1}\varphi \end{pmatrix}.$$

As forms on $c(Y)$, harmonic for $d+\delta$, these represent

$$\varphi \quad \text{and} \quad x^{-n+2k+1}\,dx \wedge \varphi\,. \tag{3.6}$$

Thus each harmonic form on Y gives rise to two harmonic forms on $c(Y)$, one in the same degree and one a degree higher.

In deciding on the domain of $d+\delta$ we shall have to decide on what combination of these null solutions, localized near $x=0$, to admit to the domain. We certainly require the functions so admitted to be square-integrable, with respect to the conic metric. Thus, consider which of these forms is in $L^2_{\mathrm{loc},g}\left(c(Y)\right)$, i.e. is locally square-integrable near 0. Assuming that $0 \neq \varphi \in H^k_{\mathrm{Ho}}(Y)$,

$$\begin{aligned}\int_0^1 \int_Y \langle\varphi,\varphi\rangle_g\,dg = \int_0^1 \int_Y x^{-2k}\langle\varphi,\varphi\rangle_h x^{n-1}\,dx\,dh \\ = \|\varphi\|^2 \int_0^1 x^{n-2k-1}\,dx\end{aligned}$$

is finite if and only if $n-2k>0$, i.e. $k<n/2$. Similarly the other form is locally square-integrable if and only if

$$\int_0^1 \int_Y \langle x^{-n+2k+1}\,dx\wedge\varphi\,, x^{-n+2k+1}\,dx\wedge\varphi\rangle dg = \|\varphi\|^2 \int_0^1 x^{-n+2k-1}\,dx < \infty,$$

i.e. $k+1 > \frac{n}{2}$ (since it *is* a k-form).

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[2]Notice that the second term is a $(k+1)$-form, not a k-form.

Since k is necessarily an integer the effects of these constraints depends on the parity of the dimension, n, of $c(Y)$. Thus if $\varphi_k \in H^k_{\mathrm{Ho}}(Y)$ then

$$\dim c(Y) \text{ even } \Rightarrow \begin{cases} \varphi_k \in L^2_{loc,g} & \text{iff} \quad k \le \frac{n}{2} - 1 \\ x^{-n+2k+1}\, dx \wedge \varphi_k \in L^2_{\mathrm{loc},g} & \text{iff} \quad k+1 \ge \frac{n}{2} \end{cases}$$

$$\dim c(Y) \text{ odd } \Rightarrow \begin{cases} \varphi_k \in L^2_{\mathrm{loc},g} & \text{iff} \quad k \le \frac{n-1}{2} \\ x^{-n+2k+1}\, dx \wedge \varphi_k \in L^2_{\mathrm{loc},g} & \text{iff} \quad k+1 \ge \frac{n+1}{2}\,. \end{cases}$$

The important distinction here is that only when $\dim c(Y)$ is odd can φ and $x^{-n+2k+1}\, dx \wedge \varphi$ be both locally square-integrable, and then only for $k = \frac{n-1}{2}$, in the middle degree for forms on the boundary.

The significance of this arithmetic can be seen from the pairing relation. Suppose $\rho \in \mathcal{C}^\infty_c([0,\infty))$ and $\rho(x) = 1$ in $x < \epsilon$ for some $\epsilon > 0$. Then for the two forms $u = \varphi$ and $v = x^{-n+2k+1}\, dx \wedge u$ associated to $\varphi \in H^k_{\mathrm{Ho}}(Y)$ the localized forms ρu and ρv are mapped into $\dot{\mathcal{C}}^\infty(c(Y))$ by $d+\delta$. Thus the pairing

$$G(u,v) = ((d+\delta)(\rho(x)u)\,,\rho(x)v) - (\rho(x)u\,,(d+\delta)(\rho(x)v)) \tag{3.7}$$

is well-defined. It is independent of ρ; in fact it can be computed from the non-degenerate form of Green's formula as

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\int_{x>\epsilon}\int_Y \left(\langle (d+\delta)(\rho u), \rho v\rangle - \langle \rho u, (d+\delta)(\rho v)\rangle\right)\, dg \\ &\qquad\qquad = -\int_Y \langle\varphi,\varphi\rangle_h\, dh\,. \end{aligned} \tag{3.8}$$

Between elements arising from orthogonal forms in $H^k_{\mathrm{Ho}}(Y)$ the corresponding pairing vanishes, as it clearly does between u and itself and v and itself.

So far we have only considered elements of the null space of $d+\delta$ arising from harmonic forms on Y. Recall that the non-zero eigenvalues of $d+\delta$ are associated to pairs of forms $\mu \in \mathcal{C}^\infty(Y\,;\Lambda^{k-1})$ and $\lambda \in \mathcal{C}^\infty(Y\,;\Lambda^k)$ satisfying

$$d\mu = \tau\lambda \quad \text{and} \quad \delta\lambda = \tau\mu\,,\tau > 0\,. \tag{3.9}$$

Then $\mu + \lambda$ is an eigenform with eigenvalue τ and $\mu - \lambda$ is an eigenform with eigenvalue $-\tau$.

As discussed above the two-dimensional space spanned by μ and λ generates a four-dimension space over $\mathcal{C}^\infty((0,\infty))$ which is invariant under $d+\delta$, with basis

$$\mu\,,\ dx \wedge \mu\,,\ \lambda\,,\ dx \wedge \lambda\,.$$

Thus there is a corresponding four-dimensional null space of $d+\delta$.[3] From the grading properties of $d+\delta$ there must be two independent solutions of the form

$$f(x)\, dx \wedge \mu + g(x)\lambda \tag{3.10}$$

and two independent solutions of the form

$$f(x)\mu + g(x)\, dx \wedge \lambda\,. \tag{3.11}$$

From the homogeneity of D these solutions are spanned by homogeneous forms. Thus in the second case we can look for values of t such that

$$D\begin{pmatrix} c_1 x^t \mu \\ c_2 x^t \lambda \end{pmatrix} = 0\,.$$

[3]Since it is a first order ordinary differential operator.

Taking into account (3.5) with μ a $(k-1)$-form and λ a k- form this becomes

$$\begin{aligned}\tau c_1 + (-t-(n-k-1))c_2 &= 0\\ (t+k-1)c_1 - \tau c_2 &= 0\end{aligned}$$

i.e. t must be an eigenvalue of

$$\begin{pmatrix} -(k-1) & \tau \\ \tau & -(n-k-1) \end{pmatrix}$$

which means that $-\tau^2 + (t+k-1)(t+n-k-1) = 0$ so

$$t = -\frac{n-2}{2} \pm \left(\tau^2 + \left(\frac{n}{2}-k\right)^2\right)^{1/2}. \tag{3.12}$$

Since $\tau \neq 0$ the corresponding eigenvector can have neither component zero. The null form is then

$$u = c_1 x^{t+k-1}\mu + c_2 x^{t+k}\, dx \wedge \lambda\,. \tag{3.13}$$

The local square-integability of these forms reduces to the finiteness of the integral

$$\int_0^1 x^{2t+n-1}\, dx$$

which is to say $t + \frac{n}{2} > 0$. From (3.12) this reduces to

$$1 \pm \tau^2 + \left(\frac{n}{2}-k\right)^{1/2} > 0\,. \tag{3.14}$$

Thus the positive square-root always gives rise to a locally square-integrable solution whereas the negative root does so only if

$$\tau^2 + \left(\frac{n}{2}-k\right)^2 < 1\,. \tag{3.15}$$

Next consider the null space of $d+\delta$ on forms as in (3.10). Since these are k-forms we look for solutions of

$$D\begin{pmatrix} d_1 x^s \lambda \\ d_2 x^s \mu \end{pmatrix} = x^{s-1}\begin{pmatrix} (\tau d_1 + (-s-(n-k))d_2)\mu \\ ((s+k)d_1 - \tau d_2)\lambda \end{pmatrix} = 0.$$

Thus the solutions correspond to the singular values of s for the matrix

$$\begin{pmatrix} \tau & -s-n+k \\ s+k & -\tau \end{pmatrix}$$

and the corresponding null vectors. That is,

$$s = -\frac{n}{2} \pm \left(\tau^2 + \left(\frac{n}{2}-k\right)^2\right)^{1/2}. \tag{3.16}$$

The corresponding null eigenforms are locally square-integrable if $2s+n>0$, i.e. in this case the positive root is always admissible but the negative root is *never* admissible.

Consider the pairing between these solutions, in the sense of (3.7). Since μ and λ are orthogonal on Y the pairing vanishes on each of the two dimensional spaces in (3.13) and those just discussed,

$$v = d_1\, x^{s+k-1}\, dx \wedge \mu + d_2 x^{s+k}\lambda\,. \tag{3.17}$$

On the other hand the pairing between solutions in (3.13) and (3.17)

$$((d+\delta)\rho u\,,\rho v)-(\rho u\,,(d+\delta)\rho v) = \left(-c_1\cdot d_1\|\mu\|^2+c_2\cdot d_2\|\lambda\|^2\right)\lim_{\epsilon\downarrow 0}\epsilon^{s+t+n-1} \tag{3.18}$$

Since the left side is finite, the pairing between solutions in (3.13) and (3.17) with the same sign of square-root must vanish[4] whereas that between opposite signs is non-degenerate. More precisely, amongst locally square-integrable solutions those with negative square-root in (3.7) pair in a non-degenerate fashion with the corresponding (i.e. the same μ and λ) solutions in (3.10) with the positive square root in (3.16). All other pairings between square-integrable solutions vanish, except for the harmonic case previously discussed.

Summarizing the computations so far we have described the tempered null space of $d+\delta$ in the model case.

Proposition 12. *If $\mu\in\mathcal{S}'([0,\infty)X,Y;\Lambda^*)$ is a solution of $(d+\delta)\mu=0$ for the conic metric on $c(Y)=[0,\infty)\times Y$ then μ is a finite linear combination of the null solutions in* (3.10), (3.11) *with powers from* (3.12) *and* (3.16).

To prove this I shall detour into a discussion of the Mellin transform, but this will actually quickly lead to a description of the fundamental solutions for $d+\delta$ — which is what we really want.

So, recall that the one-dimensional Fourier transform

$$\begin{aligned}\hat\eta(z) &= \int e^{-izt}\eta(t)\,dt\\ \eta(t) &= \frac{1}{2\pi}\int e^{izt}\hat u(z)\,dz\end{aligned}$$

has the following properties, amongst others. It gives an isomorphism (Paley-Wiener theorem)

$$\{\eta\in\mathcal{C}^\infty_c(\mathbb{R})\}\overset{\widehat{}}{\longleftrightarrow}\Big\{\hat\eta:\mathbb{C}\to\mathbb{C}\text{ entire s.t. }\exists\ A>0\\ \text{for which }\sup_{z\in\mathbb{C}}\exp(-A\,|\mathrm{Im}z|)\left|z^p\frac{d}{dz}^q\hat\eta(z)\right|<\infty\ \forall\ p,q\in\mathbb{N}_0\Big\}.$$

and, by conitnuity, an isomorphism (Plancherel theorem)

$$L^2(\mathbb{R})\ni\eta\overset{\widehat{}}{\longleftrightarrow}\hat\eta\in L^2(\mathbb{R})\,.$$

Consider the map

$$e:\mathbb{R}\ni t\mapsto x=e^{-t}\in[0,\infty)$$

which maps $\mathbb{R}$ diffeomorphically onto $(0,\infty)$. Pull-back under this map gives two isomorphisms

$$e^*:\mathcal{C}^\infty_c\left((0,\infty)\right)\leftrightarrow\mathcal{C}^\infty_c(\mathbb{R})$$

and

$$e^*:L^2_b\left([0,\infty)\right)\leftrightarrow L^2(\mathbb{R})$$

[4]Of course you may see this directly by computing the eigenvectors.

where $L^2_b([0,\infty))$ is the space of measurable functions on $[0,\infty)$ which are square-integrable with respect to the 'b-measure' dx/x, i.e

$$\int_0^\infty |v(x)|^2 \frac{dx}{x} < \infty \Leftrightarrow v \in L^2_b([0,\infty)) \,.$$

The *Mellin transform* is by definition the Fourier transform composed with this pull-back:

$$v_M(z) = \widehat{e^* v}(z) = \int_0^\infty x^{iz} v_M(z)\, dz \,.$$

The Paley-Wiener and Plancherel theorems carry over immediately to the Mellin transform to yield the basic isomorphisms

$$\mathcal{C}^\infty_c((0,\infty)) \ni v \xleftrightarrow{()_M} \Big\{ v_M : \mathbb{C} \to \mathbb{C} \text{ entire s.t.}$$
$$\text{for some } A \text{ fixed and all } p, q \sup_{z\in\mathbb{C}} \left| \exp(-A|\mathrm{Im} z|) z^p \frac{d^q}{dz} v_M(z) \right| < \infty \Big\} .$$

and

$$L^2_b([0,\infty]) \ni v \xleftrightarrow{()_M} v_M \in L^2(\mathbb{R}) \,.$$

Directly from the definition of the Fourier transform, for $\eta \in \mathcal{C}^\infty_c(\mathbb{R})$,

$$\begin{aligned} \widehat{\partial_t \eta(z)} &= i\hat{\eta}(z) \\ \widehat{e^{at}\eta(z)} &= \hat{\eta}(z + ia) \,. \end{aligned}$$

The corresponding identities for the Mellin transform are:

$$\begin{aligned} (x\partial_x v)_M(z) &= -iv_M(z) \\ (x^a v)_M &= v_M(z - ia) \,. \end{aligned}$$

Now, I shall use the Mellin transform to find a fumdamental solution for the operator $d + \delta$ on the cone. Using the decomposition (3.4) and the formula (3.5) for the resulting operator we look for a distribution $E_0(x, y, y')$, with values in $\mathrm{Hom}(\Lambda^* Y \oplus \Lambda^* Y)$, such that

$$DE_0 = \partial(x-1) Id_y \tag{3.19}$$

when Id_y is the identity operator on sections of $\Lambda^* Y \oplus \Lambda^* Y$. Let K be the number operator on forms on Y (not $c(Y)$). Then $K\varphi = k\varphi$ if $\varphi \in \mathcal{C}^\infty(Y; \Lambda^k Y)$ and the operator D has been written as

$$D = \frac{1}{x} \begin{pmatrix} d+\delta & -x\partial_x - (n-K-1) \\ x\partial_x + K & -(d+\delta) \end{pmatrix}$$

acting on sections of $\Lambda^* \oplus \Lambda^* Y$.

Equation (3.19) can then usefully be written $xDE_0 = \delta(x-1)\,\mathrm{Id}_y$. Formally taking the Mellin transform we can look instead for $E_M(z)$ satisfying

$$QE_M = \mathrm{Id}_y, \text{ where } Qv_M = (xDv)_M. \tag{3.20}$$

In fact it is more convenient to first use the ellipticity of D in $x > 0$ to choose a paramatrix for E_0, that is a distribution E'_0 with compact support in $x > 0$ (near $x = 1$) such that

$$DE'_0 = \delta(x-1)\,\mathrm{Id}_y - F(x)$$

when F is smooth with values in smoothing operators on Y and support near $x = 1$.

The use of the paramatrix to 'remove' the singular term reduces the construction to the search for a solution $G(z)$ of

$$QG(z) = (xF)_M \text{ where } \begin{pmatrix} d+\delta & iz-(n-K-1) \\ -iz+K & -(d+\delta) \end{pmatrix}. \tag{3.21}$$

Here, by the Paley-Wiener theorem, the right side is entire in z with values in the smoothing operators, with rapid decreas of all seminorms as $|\mathrm{Re} z| \to \infty$ whilst $\mathrm{Im} z$ remains bounded. Premiultiplying (3.21) by the matrix $\left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ reduces the problem to

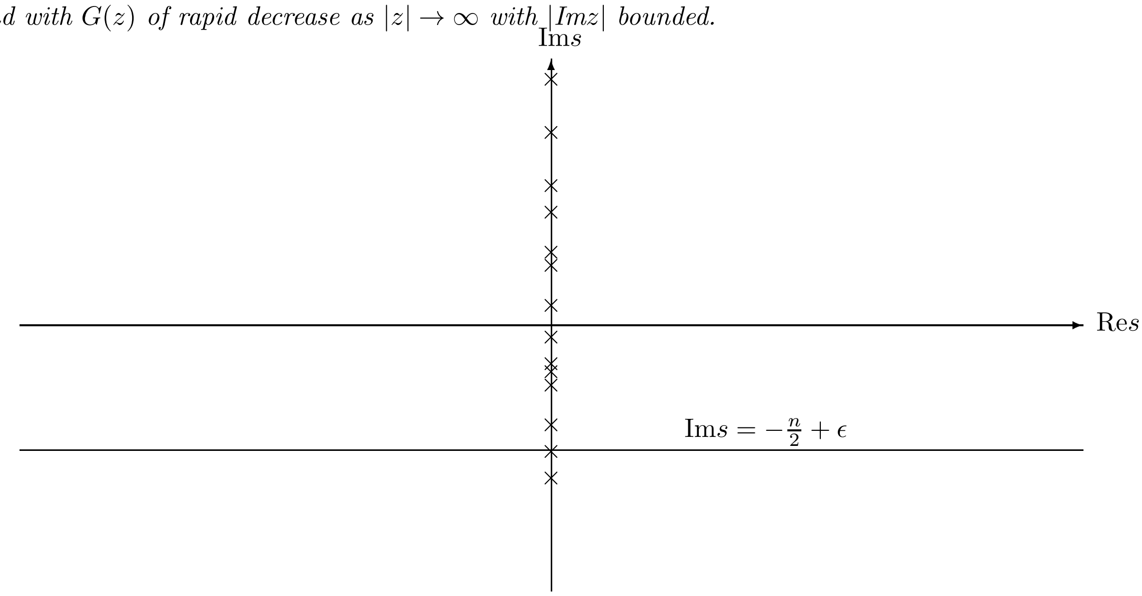
$$\begin{gathered} \tilde{Q}G(z) = (x\tilde{F})_M, \\ \tilde{Q} = \begin{pmatrix} iz-K & d+\delta \\ d+\delta & iz-(n-K-1) \end{pmatrix}, \tilde{F} = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} F. \end{gathered} \tag{3.22}$$

The singular values of $\tilde{Q}(z)$ are therefore the eigenvalues of the self-adjoint operator on $\Lambda^*Y \oplus \Lambda^*Y$

$$S = \begin{pmatrix} -K & d+\delta \\ d+\delta & -(n-K-1) \end{pmatrix}. \tag{3.23}$$

It is precisely these eigenvalues which we had previously computed. In particular the solution to (3.23) has the following properties

Lemma 7. *If F is a smooth function of x with support near $x = 1$ and values in smoothing operators on $\Lambda^*Y \oplus \Lambda^*Y$ then the solution $G(z)$ to (3.22) is a menomorphic function of z with poles only at the singular values of (3.23), on the pure imaginary axis, only of order 1, with residues finite rank smoothing operators and with $G(z)$ of rapid decrease as $|z| \to \infty$ with $|Imz|$ bounded.*



If $a \in \mathbb{R}$ is such that no singular value has imaginary part a then we can define a distribution from G by taking the inverse Mellin transform. Since the line $\mathrm{Im}s = -\frac{n}{2}$

correspond precisely to square-integrability, we shall choose

$$
\begin{aligned}
&a = -\frac{n}{2} + \epsilon, \epsilon > 0 \text{ so small that} \\
&\operatorname{Im} z \notin \left(-\frac{n}{2}, -\frac{n}{2} + 2\epsilon\right) \text{ if } \tilde{Q}(z) \text{ is singular}.
\end{aligned} \tag{3.24}
$$

Then, as a function of $s \in \mathbb{R}$

$$G(s + ia) \in L^2\left(\mathbb{R}; \mathcal{C}^\infty(Y \times Y; \operatorname{Hom}(\Lambda^* \oplus \Lambda^*)\right)$$

so we can define a kernel G'_a by

$$\left(x^{-a} G'_a\right)_M (s) = G(s + ia). \tag{3.25}$$

For this choice of contour of integration we can easily describe the asymptotic expansion of G'_a.

PROPOSITION 13. *For the choice of a in (3.24), G'_a is a smooth kernel on $(0, \infty) \times Y^2$ with a complete asymptotic expansion at $x = 0$ and $x = \infty$:*

$$G'_a \sim \sum_{\{j; s_j > -\frac{n}{2}\}} x^{s_j} e_j(y, y') \text{ as } x \downarrow 0 \tag{3.26}$$

$$G'_a \sim \sum_{\{j; s_j \leq -\frac{n}{2}\}} x^{-s_j} e_j(y, y') \text{ as } x \uparrow \infty, \tag{3.27}$$

where s_j are the constants in

2. Metric Horns

As preparation for the more general models arising in the study of stratified varieties I shall also examine the Hodge theory for a compact manifold with boundary equipped with a metric g which makes a neighbourhood of the boundary into a 'horn'. Thus, in some neighbourhood of the boundary and for some product decomposition

$$g = dx^2 + x^{2a} h \qquad 0 < x < \epsilon, \qquad 0 < a < 1. \tag{3.28}$$

In the 'limiting cases' $a = 0$ and $a = 1$ the non-degenerate and conic cases, as previously considered, appear.

Following the general procedure outlined above we shall investigate possible choices for the domain of $d + \delta$ defined with respect to such a metric and also analyze the 'model' case of a metric globally of the form (3.28) on the product $c(Y)$ for a compact manifold (without boundary for the moment) Y.

The form of the metric in (3.28) can be changed by introducing the singular boundary variable $t = cx^{1-a}$ so that

$$g = t^{2\alpha}(dt^2 + h') \qquad 0 < \alpha < \infty, \ \alpha = \frac{a}{1-a}. \tag{3.29}$$

It is slightly more convenient to work with the 'conformal' metric in (3.29) than the metric in (3.28). Note that this singular transformation changes the $\mathcal{C}^\infty$ structure on the manifold, near the boundary. However it does not change the 'conormal' structure in the sense that $\mathcal{A}_{\text{phg}}(X)$ is unaltered.

The Hodge decompositions we will obtain are for the target space

$$\operatorname{Tar}(X) = \mathcal{A}_{\text{phg}}(X; \Lambda^* X) \cap L^2_g(X; \Lambda^* X), \tag{3.30}$$

so they are not affected by change of variable. The Riemannian measure of the metric (3.28) is[5]

$$dg = t^{\alpha n}\, dt\, dh\,. \tag{3.31}$$

As usual, near the boundary or globally in the model case, it is convenient to 'rescale' the form bundles by identifying sections of $\Lambda^* X$ with sections of $\Lambda^* Y \oplus \Lambda^* Y$ where $Y = \partial X$, using

$$\Lambda^k X \ni u = t^{\alpha k}\varphi + t^{\alpha k}\, dt \wedge \psi\,. \tag{3.32}$$

Then the fibre inner product on $\Lambda^* X$ becomes

$$\langle u, u' \rangle_g = \langle \varphi, \varphi' \rangle_h + \langle \psi, \psi' \rangle_h\,.$$

The condition $u \in \mathcal{A}_{\mathrm{phg}}(X, \Lambda^* X)$ is the requirement that u be $\mathcal{C}^\infty$ in X° and have an expansion near the boundary; this expansion can be written in terms of the tangential forms φ and ψ:

$$\varphi \sim \sum_{i,k} t^{s_i'} (\log t)^k \varphi_{ik},\ \ \psi \sim \sum_{i,k} t^{s_i''} (\log t)^k \psi_{ik}. \tag{3.33}$$

Lemma 8. *The polyhomogeneous conormal target space* (3.30) *corresponds to those tangential forms with expansions* (3.33) *having*

$$\mathbb{R}e s_i'\,, \quad \mathbb{R}e s_i'' > -\frac{1}{2} - \frac{\alpha n}{2}\,. \tag{3.34}$$

Proof. The condition is the convergence of $\int_0^1 t^{2s+\alpha n}\, dt$. ☐

Since it is all that really arises below we shall assume that the exponents s_i', s_i'' are real, but this is actually irrelevant.

In terms of the decomposition (3.32), d is represented by the matrix of operators

$$t^{-\alpha} \begin{pmatrix} d & 0 \\ \partial_t + \frac{\alpha K}{t} & -d \end{pmatrix}, \quad K = k \text{ on } \Lambda^k Y\,.$$

The computation of the formal adjoint is given by

$$\begin{aligned} &\int_0^1 \int_Y t^{-\alpha} \langle d\varphi, \varphi' \rangle + t^{-\alpha} \langle \partial_t \varphi + \frac{\alpha k}{t}\varphi, \psi' \rangle - t^{-\alpha} \langle d\psi, \psi' \rangle t^{\alpha n}\, dt\, dh \\ &= \int_0^1 \int_Y t^{-\alpha} \langle \varphi, \delta\varphi' \rangle + t^{-\alpha} \langle \varphi, \left(-\partial_t - \frac{1}{t}\left(\alpha(n-1) - \alpha k\right)\right) \psi' \rangle \\ &\qquad\qquad - t^{-\alpha} \langle \psi, \delta\psi' \rangle t^{\alpha n}\, dt\, dh, \end{aligned}$$

where φ and ψ have supports in $(0, 1) \times Y$.

Thus δ is represented by the matrix:

$$t^{-\alpha} \begin{pmatrix} \delta & -\partial_t - \frac{\alpha}{t}(n - K - 1) \\ 0 & -\delta \end{pmatrix}$$

and the operator $d + \delta$ by the matrix

$$D = t^{-1-\alpha} \begin{pmatrix} t(d+\delta) & -t\partial_t - \alpha(n - K - 1) \\ t\partial_t + \alpha K & -t(d+\delta) \end{pmatrix}. \tag{3.35}$$

[5] I will drop the prime for h' in (3.29) and consider it just as some metric h on ∂X.

If we write $\mathcal{A}^{[m]}_{\mathrm{phg}}(X;\Lambda^*X)$ for the subspace of $\mathcal{A}_{\mathrm{phg}}(X;\Lambda^*X)$ consisting of the elements of 'order' at least m in the sense that the expansions in (3.33) only have terms with $\operatorname{Re} s_i' > m$ and $\operatorname{Re} s_i'' > m$ then we find

PROPOSITION 14. *For any $m \in \mathbb{R}$,*

$$d+\delta : \mathcal{A}^{[m]}_{phg}(X;\Lambda^*X)/\dot{C}^\infty(X;\Lambda^*X) \longrightarrow \mathcal{A}^{[m-1-\alpha]}_{phg}(X;\Lambda^*X)/\dot{C}^\infty(X;\Lambda^*X) \tag{3.36}$$

is surjective.

PROOF. Certainly $d+\delta$ acts on $\dot{C}^\infty(X;\Lambda^*X)$, so the map (3.36) is well defined, given the presence of the leading factor $t^{-1-\alpha}$ in (3.35), and the fact that the entries in the matrix clearly preserve order.

Since (3.36) is a map of the formal expansions, to prove its surjectivity it suffices to prove the surjectivity on the quotients

$$d+\delta : \mathcal{A}^{[m]}_{\mathrm{phg}}(X;\Lambda^*X)/\mathcal{A}^{[m+\gamma]}_{\mathrm{phg}}(X;\Lambda^*X) \to \mathcal{A}^{[m-1-\alpha]}_{\mathrm{phg}}(X;\Lambda^*X)/\mathcal{A}^{[m-1-\alpha+\gamma]}_{\mathrm{phg}}(X;\Lambda^*X) \text{ for any one } \gamma > 0\,.$$

Taking $\gamma < 1$ this simplifies the question since the diagonal terms in (3.35) increase order by 1. Thus the problem is reduced to the surjectivity of the map

$$\begin{pmatrix} 0 & -t\partial_t - \alpha(n-K-1) \\ t\partial_t + \alpha K & 0 \end{pmatrix},$$

as a map on formal expansions (3.33), with $s_i',\ s_i'' > m$. The surjectivity follows from the fact that

$$(t\partial_t + \alpha k) = t^{-\alpha k}(t\partial_t)t^{\alpha k}\,,$$

and the corresponding surjectivity for $t\partial_t$. Thus

$$t\partial_t \cdot t^s(\log t)^k\varphi = st^s(\log t)^k\varphi + kt^s(\log t)^{k-1}\varphi.$$

For $s \neq 0$, $t\partial_t$ is then an isomorphism on forms with a single power. For $s = 0$ it has null space the constants but is still surjective. □

This discussion not only shows the surjectivity, but allows the null space of (3.36) to be readily described. It is useful to refine the statement a little in terms of the Hodge decomposition of ∂X.

Observe that the quotient spaces can be viewed as tensor products:

$$\mathcal{A}^{[m]}_{\mathrm{phg}}(X;\Lambda^*X)/\mathcal{A}^{[m+\gamma]}_{\mathrm{phg}}(X;\Lambda^*X) \cong \left(\mathcal{A}^{[m]}_{\mathrm{phg}}([0,\infty))/\mathcal{A}^{[m+\gamma]}_{\mathrm{phg}}([0,\infty)\right)\otimes\left(\mathcal{C}^\infty(\partial X;\Lambda^*\partial X)\right)^2\,. \tag{3.37}$$

Indeed this is just the decomposition (3.33). We can consider various subspaces in terms of (3.37), much as in the conic case. In terms of the Hodge decomposition of $\mathcal{C}^\infty(X;\Lambda^*\partial X)$ set

$$\begin{aligned} V^{(k)} &= H^k_{\mathrm{Ho}}(\partial X)\oplus 0\,,\quad W^{(k)} = 0\oplus H^{k-1}_{\mathrm{Ho}}(X)\\ E^{(k)} &= d\mathcal{C}^\infty(\partial X;\Lambda^{k-1}\partial X)\oplus\delta\mathcal{C}^\infty(\partial X;\Lambda^k\partial X)\\ F^{(k)} &= \delta\mathcal{C}^\infty(\partial X;\Lambda^{k+1}\partial X)\oplus d\mathcal{C}^\infty(\partial X;\Lambda^k\partial X)\,. \end{aligned} \tag{3.38}$$

The Hodge decomposition then implies that the sum over k of the sum of the four subspaces

$$\begin{aligned}
\mathcal{V}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes V^{(k)} \\
\mathcal{W}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes W^{(k)} \\
\mathcal{E}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes E^{(k)} \\
\mathcal{F}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes F^{(k)} ,
\end{aligned}$$

where $\mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) = \mathcal{A}^{[m]}_{\text{phg}}([0,\infty))/\mathcal{A}^{[m+\gamma]}_{\text{phg}}([0,\infty))$, is the whole of (3.37). Furthermore, from the form of D in (3.35) we conclude that:

$$\begin{aligned}
t^{1+\alpha} D &: \mathcal{V}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{W}^{[m,\gamma]}_{(k+1)}(X) \\
t^{1+\alpha} D &: \mathcal{W}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{V}^{[m,\gamma]}_{(k-1)}(X) \\
t^{1+\alpha} D &: \mathcal{E}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{F}^{[m,\gamma]}_{(k-1)}(X) \\
t^{1+\alpha} D &: \mathcal{F}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{E}^{[m,\gamma]}_{(k+1)}(X) .
\end{aligned} \tag{3.39}$$

Since these spaces give a direct sum decomposition of the spaces in (3.37) we conclude from Proposition 14 that each of the maps in (3.39) is surjective. Since this is true for all γ we do in fact get surjectivity on the corresponding spaces of formal series.

COROLLARY 2. *The maps in* (3.39) *are surjective for all* m, γ *(including* $\gamma = \infty$*).*

From this decomposition we can easily find the null space of the map in (3.36).

PROPOSITION 15. *For* $m \ll 0$ *the null space of* $d + \delta$ *as a map on formal expressions in* (3.36) *can be identified with* $\mathcal{C}^\infty(\partial X; \Lambda^* \partial X) \oplus \mathcal{C}^\infty(\partial X; \Lambda^* \partial X)$ *where the map from this 'generalized boundary data' to formal solutions can be written in terms of* (3.38) *as:*

$$\begin{aligned}
V^{(k)} \ni \begin{pmatrix} \varphi \\ 0 \end{pmatrix} &\longmapsto \begin{pmatrix} t^{-\alpha k}\ \varphi \\ 0 \end{pmatrix} \in \mathcal{V}^*_{(k)} \\
W^{(k)} \ni \begin{pmatrix} 0 \\ \psi \end{pmatrix} &\longmapsto \begin{pmatrix} 0 \\ t^{-\alpha(n-k)}\ \psi \end{pmatrix} \in \mathcal{W}^*_{(k)} \\
E^{(k)} \ni \begin{pmatrix} d\varphi \\ 0 \end{pmatrix} &\longmapsto \begin{pmatrix} t^{-\alpha k}\ d\varphi \\ 0 \end{pmatrix} + u'(d\varphi) \in \mathcal{E}^*_{(k)} \\
E^{(k)} \ni \begin{pmatrix} 0 \\ \delta\psi \end{pmatrix} &\longmapsto \begin{pmatrix} 0 \\ t^{-\alpha(n-k)}\ \delta\psi \end{pmatrix} + u''(\delta\psi) \in \mathcal{E}^*_{(k)} \\
F^{(k)} \ni \begin{pmatrix} \delta\varphi \\ 0 \end{pmatrix} &\longmapsto \begin{pmatrix} t^{-\alpha k}\ \delta\varphi \\ 0 \end{pmatrix} + u'(\delta\varphi) \in \mathcal{F}^*_{(k)} \\
F^{(k)} \ni \begin{pmatrix} 0 \\ d\psi \end{pmatrix} &\longmapsto \begin{pmatrix} 0 \\ t^{-\alpha(n-k-2)}\ d\psi \end{pmatrix} + u''\,(d\psi) \in \mathcal{F}^*_{(k)}
\end{aligned} \tag{3.40}$$

where in the last four cases, the extra terms vanish to one order higher than the leading terms.

PROOF. This just reduces to the analysis of the null space of $t^{1+\alpha}D$. This operator is the sum

$$t^{1+\alpha}D = \begin{pmatrix} 0 & -t\partial_t - \alpha(n-K-1) \\ t\partial_t + \alpha K & 0 \end{pmatrix} + t \begin{pmatrix} d+\delta & 0 \\ 0 & -(d+\delta) \end{pmatrix} \tag{3.41}$$

where the first term preserves boundary order and the second raises it by one (at least). The first two forms are annihilated by D. The leading parts of the remaining four forms are annihilated by the first term in (3.41). Thus, by the surjectivity in the corollary above there are solutions of the form indicated.

Conversely, these span all solutions since the leading part of $t^{1+\alpha}D$ in (3.41) must annihilate the leading term of a formal solution and all such terms appear in the list. □

Proposition 15 does not quite give a natural isomorphism, although one can be chosen; the choice does not affect the important Green pairing.

PROPOSITION 16. *If the isomorphism of Proposition 15 from the space consisting of two copies of*

$$\mathcal{C}^\infty(\partial X; \Lambda^* \partial X)$$

to formal null solutions is written $(R \oplus A)^{-1}$ *then for any pair of formal solutions,* u, v

$$\begin{aligned} G(u,v) &= ((d+\delta)(\rho\tilde{u}), \rho\tilde{v}) - (\rho\tilde{u}, (d+\delta)\rho\tilde{v}) \\ &= -(Ru, Av) + (Au, Rv). \end{aligned} \tag{3.42}$$

PROOF. Here, if u and v are formal solutions, $\tilde{u}$ and $\tilde{v}$ are finite sums consisting of all terms up to a sufficiently high order and $\rho \in \mathcal{C}_c^\infty([0,\infty))$ is identically equal to 1 near $t = 0$. The pairing in (3.42) is then well-defined. Furthermore, by the pairing relation in the smooth and non-degenerate case,

$$G(u,v) = \lim_{\epsilon \downarrow 0} -(R_\epsilon \tilde{u}, A_\epsilon \tilde{v})_{h_\epsilon} + (A_\epsilon \tilde{u}, R_\epsilon v)_{h_\epsilon}$$

where R_ϵ is pull-back to $t = \epsilon$ and $A_\epsilon = R_\epsilon \cdot \iota_\nu$ where the unit normal vector field is $t^{-\alpha}\delta_t$.

The orthogonality between the various spaces in (3.38) shows that G vanishes except in pairs arising from $V^{(k)} \times W^{(k+1)}$, $E^{(k)} \times F^{(k-1)}$ (in either order since G is always antisymmetric). For these cases working directly from the leading terms given by Proposition 15 we see that $G(u,v)$ is given by (3.42). □

Recall that we are taking the target space to be

$$\mathrm{Tar}(X) = \mathcal{A}_{\mathrm{phg}}^{[-\frac{1}{2}-\frac{\alpha n}{2}]}(X; \Lambda^* X), \tag{3.43}$$

just the intersection of the polyhomogeneous conormal space with $L^2_g(X; \Lambda^* X)$. From the definition of $\mathrm{Max}(X)$, in (2.26), and the properties of D we have

$$\mathrm{Max}(X) \supset t^{1+\alpha}\,\mathrm{Tar}(X) = \mathcal{A}_{\mathrm{phg}}^{[\frac{1}{2}-\frac{(n-1)\alpha}{2}]}(X; \Lambda^* X). \tag{3.44}$$

We proceed to examine the quotient, which consists of finite expansions in the corresponding range of 'width' $1+\alpha$:

$$Q(X) = \mathrm{Max}(X)/t^{1+\alpha}\,\mathrm{Tar}(X) \subset \mathcal{A}_{\mathrm{phg}}^{[-\frac{1}{2}-\frac{n\alpha}{2}, 1+\alpha]}(X, \Lambda^* X). \tag{3.45}$$

Now each $[u] \in Q(X)$ has a representative $u \in \mathrm{Tar}(X)$ with du, $\delta u \in \mathrm{Tar}(X)$. The surjectivity in Proposition"14" means that $[u] = [u']$ where u' has $(d+\delta)u' \equiv 0$, as well as $du', \delta u' \in \mathrm{Tar}(X)$. Thus the terms in Q are determined by the formal solutions of $(d+\delta)u = v$:

$$Q(X) \cong \{[u]; u \in \mathrm{Tar}(X), (d+\delta)u \equiv 0,\, du, \delta u \in \mathrm{Tar}(X)\} \tag{3.46}$$

where the equivalence relation is always modulo $t^{1+\alpha}\,\mathrm{Tar}(X)$.

The three classes of solutions, $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{E}$, are all form-graded, so $du \equiv 0$ and $\delta u \equiv$ separately; the last condition in (3.46) therefore only arises for solutions in $\mathcal{F}$. The leading term in the k-form component if any formal null form is either $t^{-\alpha k}$ or $t^{-\alpha(n-k)}$ and it is correspondingly tangential or normal. Such terms are

$$\text{square-integrable iff } \begin{cases} k < \frac{n}{2} + \frac{1}{2\alpha} \\ k > \frac{n}{2} - \frac{1}{2\alpha}. \end{cases} \tag{3.47}$$

$$\text{int} t^{1+\alpha}\,\mathrm{Tar}(X) \text{ iff } \begin{cases} k < \frac{n}{2} - 1 - \frac{1}{2\alpha} \\ k > \frac{n}{2} + 1 + \frac{1}{2\alpha}. \end{cases} \tag{3.48}$$

Thus we only need to examine terms with leading tangential k-form parts for $\frac{n}{2} - 1 - \frac{1}{2\alpha} \le h < \frac{n}{2} + \frac{1}{2\alpha}$ and with leading normal parts for $\frac{n}{2} - \frac{1}{2\alpha} < k \le \frac{n}{2} + 1 + \frac{1}{2\alpha}$; since outside these ranges the corresponding terms are either not square-integrable or else are zero in the quotient.

Lemma 9. *"Q" The quotient $Q(X)$ in ((3.46)) consists precisely of the span of the terms in $\mathcal{V}_{(k)}$ and $\mathcal{W}_{(k)}$ for the ranges*

$$\frac{n}{2} - 1 - \frac{1}{2\alpha} \le k < \frac{n}{2} + \frac{1}{2\alpha} \tag{3.49}$$

$$\frac{n}{2} - \frac{1}{2\alpha} < k \le \frac{n}{2} + 1 + \frac{1}{2\alpha}, \tag{3.50}$$

terms in $\mathcal{E}$ with leading tangential and normal parts for the same ranges and terms in $\mathcal{F}$ with leading tangential and normal parts being k-forms for the ranges.

$$\frac{n}{2} - 1 - \frac{1}{2\alpha} \le k < \frac{n}{2} - 1 + \frac{1}{2\alpha} \tag{3.51}$$

$$\frac{n}{2} + 1 - \frac{1}{2\alpha} < k \le \frac{n}{2} + 1 + \frac{1}{2\alpha}. \tag{3.52}$$

Proof. The k-forms in $\mathcal{F}$ are never graded. Since $Q(X)$ must be graded, these are only admissible if the second term in the expansion, which has precisely one additional factor of t, is in $t^{1+\alpha}\,\mathrm{Tar}(X)$. This gives the altered (and shortened) ranges in (3.51), (3.52). □

Next we compute the boundary pairing on $Q(X)$. That is,

$$B(u,v) = (d\tilde{u}, \tilde{v}) - (\tilde{u}, \delta\tilde{v}) \tag{3.53}$$

where $\tilde{u}, \tilde{v}$ are representatives of $u, v \in Q(X)$. Here we are using the observation that the pairing vanishes on $t^{1+\alpha}\,\mathrm{Tar}(X)$, so descends to $Q(X)$. We can see this at the same time as computing B since

$$B(u,v) = \lim_{\epsilon\downarrow 0}\int_{t>\epsilon} \left(\langle d\tilde{u},\tilde{v}\rangle_g - \langle \tilde{u},\delta\tilde{v}\rangle_g\right)\,dg$$
$$= \lim_{\epsilon\downarrow 0}\int_{\partial X} \left(i_\epsilon^*\tilde{u}, i_\epsilon^*\subset \nu_\epsilon\tilde{u}\right)_{h_\epsilon}\,dh_\epsilon\,, \tag{3.54}$$

using the non-degenerate boundary pairing. Thus $dh_\epsilon = \epsilon^{(n-1)\alpha}\,dh$, $\nu_\epsilon = \epsilon^{-\alpha}\,\partial_t$ is the unit inward-pointing normal and the inner product for h_ϵ scaled away by (3.32).

In terms of the tangent-normal decomposition we therefore conclude that

$$B(u,v) = -\sum (\varphi'_{i'k}, \psi''_{i''k})_h \tag{3.55}$$

where that sum is over those terms satisfying $s_{i'} + s''_{i''} = (n-1)\alpha$. Thus we find that B vanishes except on the span of

$$\mathcal{V}_{(k)}\times\mathcal{W}_{(k+1)}\,,\mathcal{E}_{(k)}\times\mathcal{F}_{(k-1)}\ \text{ and }\mathcal{E}_{(k-1)}\times\mathcal{F}_{(k)}\,.$$

Combining these various observations we arrive at the following descriptions of $Q(X)$.

PROPOSITION 17. *"RA" If the identification of the formal null space of $d+\delta$ with $\mathcal{C}^\infty(X;\Lambda^*X)\oplus\mathcal{C}^\infty(\partial X;\lambda^*\partial X)$ given by Proposition 15" is written $R\oplus A$ then B is identified with the boundary pairing*

$$B(u,v) = -(Ru,Av)_h \tag{3.56}$$

3. Cusplike metrics

The next class of metrics we consider on a general manifold with boundary has the same form as (3.28) but are 'on the other side' of the conic metrics:

$$g = dx^2 + x^{2a}h\,,\ 1 < a < \infty\,. \tag{3.57}$$

As in the case of metric horns these can be reduced to a 'variable' conformal multiple of a fixed metric. Namely setting $t = cx^{a-1}$, for an appropriate constant c, gives

$$g = t^{2\beta}\left(\frac{dt^2}{t^4} + h'\right)\,,1<\beta<\infty\,,\ \beta = \frac{a}{a-1}\,. \tag{3.58}$$

Notice that this is really the same as in the horn case, except that t has to be replaced by $1/t$ so that the boundary is still at $t=0$ (instead of $t=\infty$).

Now, the identification for forms is analogous to (3.32), except that dt/t^2 has length $t^{-\beta}$, so we decompose the form bundle as

$$\Lambda^k X \ni u = t^{\beta k}\varphi + t^{\beta k}\frac{dt}{t^2}\wedge\psi\,. \tag{3.59}$$

Again this reduces the pointwise inner product to

$$\langle u,u'\rangle_g = \langle\varphi,\varphi'\rangle_h + \langle\psi,\psi'\rangle_h\,. \tag{3.60}$$

The volume form is $dg = t^{\beta n}\,dt\,dh/t^2$; again I will drop the prime on h.

LEMMA 10. *The polyhomogeneous conormal target space*

$$\mathrm{Tar}(X) = \mathcal{A}_{phg}(X;\Lambda^*X)\cap L^2_g(X;\Lambda^*X) \tag{3.61}$$

corresponds to those tangential forms with expansions (3.33) having

$$\mathbb{R}\mathrm{e} s_i',\ \mathbb{R}\mathrm{e} s_i'' > \frac{1}{2} - \frac{\beta n}{2}\,. \tag{3.62}$$

Proof. The condition has changed to $\int_0^1 t^{2s+\beta n}\frac{dt}{t^2} < \infty$. ☐

From the same sort of computations as before d is represented by

$$t^{-\beta}\begin{pmatrix} d & 0 \\ t^2\partial_t + t\beta K & -d \end{pmatrix}.$$

Similarly δ acts as

$$t^{-\beta}\begin{pmatrix} \delta & t^2\partial_t - \beta t(n-K-1) \\ 0 & -\delta \end{pmatrix}.$$

Thus $d+\delta$ becomes

$$D = t^{-\beta}\begin{pmatrix} d+\delta & -t^2\partial_t - \beta t(n-K-1) \\ t^2\partial_t + t\beta K & -(d+\delta) \end{pmatrix} \tag{3.63}$$

where, as before, K is the number operator on forms on the boundary.

We get an analogue of Proposition 14, but a little care is necessary.

Proposition 18. *The operator D in (3.63) defines an isomorphism for any m*

$$\begin{aligned} D : \mathcal{A}_{phg}^{[m,\infty)}\left([0,\infty); (H^*_{\mathrm{Ho}}(Y))^{\perp} \oplus (H^*_{Ho}(Y))^{\perp}\right) \to \\ \mathcal{A}_{phg}^{[m-\beta,\infty)}\left([0,\infty); (H^*_{\mathrm{Ho}}(Y))^{\perp}\right) \oplus (H^*_{\mathrm{Ho}}(Y))^{\perp} \end{aligned} \tag{3.64}$$

and surjective map

$$\begin{aligned} D : \mathcal{A}_{phg}^{[m,\infty)}([0,\infty)) \otimes (H^*_{\mathrm{Ho}}(Y) \oplus H^*_{\mathrm{Ho}}(Y)) \to \\ \mathcal{A}_{phg}^{[m-\beta+1,\infty)}([0,\infty)) \otimes (H^*_{\mathrm{Ho}}(Y) \oplus H^*_{\mathrm{Ho}}(Y))\,. \end{aligned} \tag{3.65}$$

Proof. In (3.64), D is acting on formal forms with values orthogonal to the null space of $H^*_{\mathrm{Ho}}(Y)$, in both factors, so certainly gives a map as indicated. To prove that (3.64) is an isomorphism it suffices to show that the quotient map

$$t^{\beta}D \text{ is an isomorphism on } \mathcal{A}_{\mathrm{phg}}^{[m,\gamma]}\left([0,\infty); (H^*_{\mathrm{Ho}}(Y) \oplus H^*_{\mathrm{Ho}}(Y))^{\perp}\right), \tag{3.66}$$

for any $\gamma > 0$. Taking $\gamma < 1$ the off-diagonal terms in (3.63) disappear, so (3.66) follows from the fact that $d+\delta$, the tangential operator, is an isomorphism on $(H^*_{\mathrm{Ho}}(y))^{\perp}$.

Similarly, in (3.65), the diagonal terms in (3.63) disappear and the off-diagonal terms (which vanish to one order higher) give an isomorphism as discussed in the proof of Proposition 14. ☐

Notice how different this is from the horn case. There are almost *no* formul solutions.

Proposition 19. *For a cusp-like metric the null space of D, for $m \ll 0$, consists only of the finite dimensional space spanned by*

$$t^{-\beta k}\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \text{ and } t^{-\beta(n-k-1)}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \text{ for } \varphi \in H^k_{\mathrm{Ho}}(Y)\,. \tag{3.67}$$

Proof. This is just the null space of D in (3.65), for $m \ll 0$. ☐

Applying (3.62) these formal solutions are locally square integrable when

$$\begin{cases} k < \frac{n}{2} - \frac{1}{2\beta} & \text{for } t^{-\beta k}\binom{\varphi}{0} \text{ and} \\ k+1 > \frac{n}{2} + \frac{1}{2\beta} & \text{for } t^{-\beta(n-k-1)}\binom{0}{\varphi} \end{cases} \text{ where } \varphi \in H^k_{\mathrm{Ho}}(Y)\,. \tag{3.68}$$

Writing the isomorphism on formal solutions given by (3.67) as $(R \oplus A)^{-1}$ we see, as before, that the Green pairing on formal solutions is

$$\begin{aligned} G(u,v) &= ((d+\delta)\rho\tilde{u}, \rho\tilde{v}) - (\rho\tilde{u}, (d+\delta)\rho\tilde{v}) \\ &= -(Ru, Av)_h + (Au, Rv)_h\,. \end{aligned} \tag{3.69}$$

Proposition 20. *For a cusplike metric with* $\mathrm{Tar}(X)$ *given by (3.61),*

$$(R, A) : \mathrm{Max}(X)/\,\mathrm{Min}(X) \cong H^{\frac{n-1}{2}}_{\mathrm{Ho}}(X) \oplus H^{\frac{n-1}{2}}_{\mathrm{Ho}}(X)$$

with Green's pairing being the symplectic pairing,

$$G(u,v) = -(Ru, Av) + (Au, Rv)\,. \tag{3.70}$$

Proof. From (3.68) the only 'overlap' between the two classes of harmonic forms in (3.67) is when

$$-\frac{1}{2a} < k - \frac{n-1}{2} < 1/2a\,. \tag{3.71}$$

Since $1 < a < \infty$ this can only occur when $k = \frac{n-1}{2}$. The computation of the pairing, in (3.70), follows from(3.69). $\square$

The possibilities for the choice of the domain of $d + \delta$, consistent with the desire for a Hodge decomposition, are therefore the same as in the conic case. In particular if n is even there is no choice.