CHAPTER 1

Hodge theory of compact $\mathcal{C}^\infty$ manifolds

To start[1] with I shall briefly recall the case of a compact manifold without boundary. Since I assume that this standard case is already familiar to you, I shall discuss it here mainly to set the agenda for later.

Let X be a compact manifold with boundary. The tangent bundle TX and its dual T^*X are both well-defined manifolds which have the structure of vector bundles over X. A Riemannian metric on X is a smoothly-varying positive-definite Euclidean product on the fibres of TX. For the most part we assume that such a metric, g, is given.

The exterior powers of T^*X are the form bundles $\Lambda^k X$ with fibres $\Lambda^k_x X = \Lambda^k(T^*_x X)$; their sections are forms on X. The fundamental operator we consider is the de Rham differential. This is a continuous linear operator

$$d : \mathcal{C}^\infty(X;\Lambda^k) \longrightarrow \mathcal{C}^\infty(X;\Lambda^{k+1})\ \forall\ k.$$

In local coordinates[2] $y_1, \dots, y_n$ in a coordinate neighbourhood O of X, $n = \dim X$, $T^*_x X$ for $x \in O$ has the coordinate basis $dy_1, \dots, dy_n$. This induces a basis of each of the $\Lambda^k_x X$; namely the

$$dy^\alpha = dy_{\alpha_1} \wedge dy_{\alpha_2} \wedge \dots \wedge dy_{\alpha_k}$$

give a basis of Λ^k_x as α runs over the strictly increasing subsets[3] of $\{1, \dots, n\}$ with k elements. In terms of this basis an element $\phi \in \mathcal{C}^\infty(O, \Lambda^k)$ can be written

$$\phi = \sum_{|\alpha|=k} \phi_\alpha(y)\, dy^\alpha$$

and then

$$d\phi = \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \phi_\alpha}{\partial y_j}\, dy_j \wedge\, dy^\alpha$$

$$dy_j \wedge\, dy^\alpha = \begin{cases} 0 & \text{if } j \in \alpha \\ (-1)^{\ell(\alpha,j)}\, dy^\beta & \text{if } j \notin \alpha,\ \beta = \alpha \cup \{j\} \end{cases}$$

with $l(\alpha, j)$ the number of elements of α less than j.

The fundamental property of d is that $d^2 = 0$ so we get the de Rham complex

$$\mathcal{C}^\infty(X) \xrightarrow{d} \mathcal{C}^\infty(X;T^*X) \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^2 X) \xrightarrow{d} \cdots$$
$$\cdots \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^n) \xrightarrow{d} 0.$$

[1]Thanks To Victor Nistor for helpful comments on this chapter.

[2]Thus there is a diffeomorphism $O \xleftrightarrow{F} O' \subset \mathbb{R}^n$ onto an open set such that the y_j on the pull-backs under F of the standard coordinates on $\mathbb{R}^n$.

[3]Of course I am really taking α to be an increasing *map* $\alpha : \{1, \dots, k\} \longrightarrow \{1, \dots, n\}$.

The homology of this complex is the ($\mathcal{C}^\infty$) de Rham cohomology of X

$$H^k_{\mathrm{dR}}(X) = \frac{\{u \in \mathcal{C}^\infty(X;\Lambda^k); du = 0\}}{d\mathcal{C}^\infty(X;\Lambda^{k-1})}.$$

THEOREM 1. *(De Rham) For a compact $\mathcal{C}^\infty$ manifold $H^k_{\mathrm{dR}}(X)$ has a natural non-degenerate pairing with $H_{k,\Delta}(X;\mathbb{R})$ the simplicial (or the singular, or the Čech ...) homology with real coefficients:*

$$\langle u,\sigma\rangle = \sum_i c_i \int_{\Delta_k} m^*_{\sigma_1} u,\ \sigma = \sum_i c_i m_i\,.$$

The space of square-integrable sections of a vector bundle on X is also well-defined by reference to local coordinates. Then d can be extended to a densely-defined operator on L^2 by setting

$$\mathrm{Dom}^k(d) = \{u \in L^2(X;\Lambda^k);\ du \in L^2(X;\Lambda^{k+1})\}$$

where the second condition can be taken in either a strong or a weak sense. Given such a definition, the L^2 cohomology of X is

$$H^k_{L^2}(X) = \frac{\left\{u \in \mathrm{Dom}^k(X); du = 0\right\}}{d\,\mathrm{Dom}^{k-1}(X)} \cong H^k_{\mathrm{dR}}(X),$$

another form of the de Rham theorem.

The main thrust of these lectures is about Hodge theory. This uses the choice of a metric g. By definition this is a Euclidean inner product on the fibres of TX; by duality it fixes a Euclidean inner product in the fibres of T^*X. Let $e_1, \dots, e_n$ be an orthonormal basis of T^*_xX. The assertion that the $e^\alpha = e_{\alpha_1} \wedge e_{\alpha_2} \wedge \ldots \wedge e_{\alpha_k}$ form an orthonormal basis of Λ^k_x fixes a Euclidean inner product on Λ^k_xX. It is independent of the choice of orthonormal basis of T^*_xX. Thus the Λ^kX also have, smoothly varying, Euclidean inner products on their fibres. I shall denote these inner products by $\langle \bullet, \bullet\rangle$. The density bundle on X also has a natural smooth section fixed by the metric; in terms of any orthonormal basis its value is

$$|e_1 \wedge \cdots \wedge e_n| \in \Omega_n X.$$

I shall denote this Riemannian density dg.

The fibre inner product fixes a global inner product on $\mathcal{C}^\infty(X;\Lambda^k)$:

$$(\phi,\phi') = \int_X \langle \phi,\phi'\rangle\, dg_x\,.$$

This in turn determines a second differential operator

$$\delta : \mathcal{C}^\infty(X;\Lambda^k) \to \mathcal{C}^\infty(X;\Lambda^{k-1})$$

for each k, by

$$(du, v) = (u, \delta v)\ \forall\ u \in \mathcal{C}^\infty(X;\Lambda^{k-1}),\ v \in \mathcal{C}^\infty(X;\Lambda^k).$$

THEOREM 2. *(Hodge decomposition) The operator $d+\delta$ is Fredholm and self-adjoint on $\mathcal{C}^\infty(X;\Lambda^*)$; if $E : \mathcal{C}^\infty(X;\Lambda^*) \longrightarrow \mathcal{C}^\infty(X;\Lambda^*)$ is the unique generalized inverse which has range in the orthocomplement and annihilates the null space, $H^*_{\mathrm{Ho}}(X)$, of $d+\delta$ then*

$$\mathcal{C}^\infty(X;\Lambda^k) = dE\mathcal{C}^\infty(X;\Lambda^k) \oplus \delta E\mathcal{C}^\infty(X;\Lambda^k) \oplus H^k_{\mathrm{Ho}}(X)$$

is the decomposition into exact, coexact and harmonic k-forms.

COROLLARY 1. $H^k_{\mathrm{Ho}}(X) \cong H^k_{\mathrm{dR}}(X)$.

It is this form of the 'Hodge theorem' that I wish to generalize to various 'singular' situations. Let me briefly review the proof since this should serve to motivate some of the later enterprises. The *proof* can be reduced to three steps leading to the construction of E.

1. Construct a 'crude' parametrix E' for $d+\delta$, such that $(d+\delta)\circ E' - \mathrm{Id}$ and $E'\circ(d+\delta) - \mathrm{Id}$ are both compact operators on $L^2(X;\Lambda^*X)$.
2. Refine the parametrix to get an operator E'' such that $(d+\delta)\circ E'' - \mathrm{Id}$, $E''\circ(d+\delta) - \mathrm{Id}$ are both smoothing operators on $\mathcal{C}^\infty(X;\Lambda^*X)$.
3. Conclude that $d+\delta$ is Fredholm on $\mathcal{C}^\infty(X;\Lambda^*X)$ and that $E - E''$ is a smoothing operator.

To carry out these three steps it is very convenient to use the algebra of pseudo-differential operators $\Psi^{\mathbb{Z}}(X;\Lambda^*X)$. This consists of operators on $\mathcal{C}^\infty(X;\Lambda^*X)$. The filtration by order has a simple corresponding graded quotient i.e. there is a short exact sequence for each $k\in\mathbb{Z}$

$$0 \longrightarrow \Psi^{k-1}(X;\Lambda^*X) \longrightarrow \Psi^k(X;\Lambda^*X) \xrightarrow{\sigma_k} \mathcal{C}^\infty(S^*X; L^k\otimes \mathrm{Hom}(\Lambda^*X)) \to 0. \tag{1.1}$$

Here L is the (trivial) line bundle over $S^*X = (T^*X\setminus 0)/\mathbb{R}^+$ corresponding to functions on $T^*X\setminus 0$ which are homogeneous of degree 1.[4] This sequence is multiplicative

$$\sigma_{k+l}(AB) = \sigma_k(A)\cdot\sigma_l(B),\ \forall\ A\in\Psi^k(X;\Lambda^*X),\ B\in\Psi^l(X;\Lambda^*X).$$

Furthermore, differential operators *are* pseudo-differential operators:

$$d+\delta\in\mathrm{Diff}^1(X;\Lambda^*X)\subset\Psi^1(X;\Lambda^*X).$$

The *symbol* of a differential operator is easily computed and in this case

$$\begin{aligned}\sigma_1(d) &= i\xi\wedge = i\,\mathrm{ext}(\xi)\in\mathcal{C}^\infty(S^*X; L\otimes\mathrm{hom}(\Lambda^*X))\\ \sigma_1(\delta) &= (i\xi\lambda)^* = -i\,\mathrm{int}(\xi)\in\mathcal{C}^\infty(S^*X;L\otimes\mathrm{hom}(\Lambda^*X)).\end{aligned}$$

Here $\mathrm{int}(\xi)$, for $\xi\in T^*X$ is the action by inner product.[5]

Now, it follows from this that $\sigma_1(d+\delta)$ is an *invertible* element of $\mathcal{C}^\infty(S^*X; L\otimes \mathrm{hom}(\Lambda^*))$, with inverse in $\mathcal{C}^\infty(S^*X;L^{-1}\otimes\mathrm{hom}(\Lambda^*))$. The exactness of the sequence (1.1) means there exists

$$E'\in\Psi^{-1}(X;\Lambda^*)\text{ with }\sigma_{-1}(E') = \sigma_1(d+\delta))^{-1}.$$

This operator has the desired properties if we observe that

PROPOSITION 1. *Elements of the space $\Psi^{-1}(X;\Lambda^*X)$ define compact operators on $L^2(X;\Lambda^*X)$.*

[4] The dual metric defines (or is) a function which is positive and homogeneous of degree 2 on $T^*X\setminus 0$. This gives a trivialization of L.

[5] This is determined by the metric and contraction. If e_i is an orthonormal basis of T^*_xX with $e_1 = \xi/|\xi|$ then

$$\mathrm{int}(\xi)e^\alpha = \begin{cases}0 & \alpha_1\neq 0\\ |\xi|e^{\alpha'}, & \alpha=(\alpha_1,\alpha').\end{cases}$$

It is independent of choice of basis.

To proceed to the second step of the construction consider the 'remainder'

$$(d+\delta)\circ E' = \operatorname{Id} - R_1, \quad R_1 \in \Psi^{-1}(X;\Lambda^* X).$$

The formal Neumann series for the inverse of the right side

$$\operatorname{Id} + R_1 + R_1^2 + \cdots + R_1^k + \cdots$$

has kth term in $\Psi^{-k}(X;\Lambda^*)$. Whilst it will not converge in general we can apply the following form of 'Borel's lemma' to asymptotically sum the series.

PROPOSITION 2. *(Asymptotic completeness) For any series $\sum_j A_j$ with terms $A_j \in \Psi^{-j}(X;\Lambda^*)$ there exists $A \in \Psi^{\mathbb{Z}}(X;\Lambda^*)$ such that $A - \sum_{j<k} A_j \in \Psi^{-k}(X;\Lambda^*)$ for* all k; *furthermore A is unique modulo a term in*

$$\Psi^{-\infty}(X;\Lambda^*) = \bigcap_k \Psi^k(X;\Lambda^*).$$

The 'residual terms' here are easily understood.

PROPOSITION 3. *The two-sided ideal $\Psi^{-\infty}(X;\Lambda^*) \subset \Psi^{\mathbb{Z}}(X;\Lambda^*)$ consists precisely of the smoothing operators, meaning those operators on $\mathcal{C}^\infty(X;\Lambda^*)$ which can be written as integral operators with kernel $A \in \mathcal{C}^\infty(X^2;\hom(\Lambda^*)\otimes\Omega_{\mathbb{R}})$*

$$Au(x) = \int_X A(x,y)u(y)\,.$$

Applying the proposition on asymptotic completeness gives an element $S \in \Psi^{-1}(X;\Lambda^*)$ such that

$$(\operatorname{Id} - R_1)\circ(\operatorname{Id} + S) = \operatorname{Id} + R_2, R_2 \in \Psi^{-\infty}(X;\Lambda^*)\,.$$

Then $E'' = E'\circ(\operatorname{Id} - S)$ is an operator with the properties stated. That $(d+\delta)\cdot E'' - \operatorname{Id}$ is a smoothing operator is immediate by construction. Similarly $(\operatorname{Id} - S)\circ E''$ is a *left parametrix*, i.e. an inverse in $\Psi^{\mathbb{Z}}(X;\Lambda^*)/\Psi^{-\infty}(X;\Lambda^*)$. Since this is a ring, left and right inverses coincide so E'' is *also* a left parametrix.

At this point it has been established that $d+\delta : \mathcal{C}^\infty(X;\Lambda^*) \longrightarrow \mathcal{C}^\infty(X;\Lambda^*)$ is Fredholm, i.e. has a finite dimensional null space, denoted $H^*_{\mathrm{Ho}}(X)$, and range of finite codimension (hence closed). The self-adjointness means that $H^*_{\mathrm{Ho}}(X)$ must be a complement to the range. Hence there is a unique operator E which has null space $H^*_{\mathrm{Ho}}(X)$, range $H^*_{\mathrm{Ho}}(X)^\perp$ and inverts $d+\delta$ on its range.

It remains only to show that $E - E'' \in \Psi^{-\infty}(X;\Lambda^*)$, and hence to see that $E \in \Psi^{-1}(X;\Lambda^*)$. This follows from the 'bi-ideal' property of $\Psi^{-\infty}(X;\Lambda^*)$.

PROPOSITION 4. *If $A : \mathcal{C}^\infty(X;\Lambda^*) \longrightarrow \mathcal{C}^{-\infty}(X;\Lambda^*)$ (distributional sections) is any continuous linear operator and P, $Q \in \Psi^{-\infty}(X;\Lambda^*)$ then $PAQ \in \Psi^{-\infty}(X;\Lambda^*)$.*

Thus E and E' satisfy

$$PE - \operatorname{Id} = Q,\ PE' - \operatorname{Id} = Q' \text{ and}$$
$$EP - \operatorname{Id} = R,\ E'P - \operatorname{Id} = R'$$

with $Q, Q', R, R' \in \Psi^{-\infty}(X;\Lambda^* X)$. Combining the first and last of these equations gives

$$E - E' = E'Q - R'E.$$

Similarly combining the other two gives

$$E - E' = E'R - EQ'.$$

Inserting the second of these in the first gives

$$E - E' = E'Q - R'E' - R'E'Q + R'EQ'.$$

All terms on the right are in $\Psi^{-\infty}(X;\Lambda^* X)$, the last by Proposition 4.

Now, my objective below is to give a similar Hodge decomposition, and a similar proof of it, for certain singular spaces. With this in mind let us observe that the three steps in the proof — which is the construction of E — have the following flavour. First, we solve a *model problem.* In this case it is the inversion of $\sigma_1(d+\delta)$; this is closely related to the Hodge theory of Euclidean space. Secondly an iterative argument allows the 'error' to be refined so that we can discuss rather precisely the mapping properties of $d+\delta$. The third (and somewhat optional) step is to show that the 'true inverse' E is actually very close to the approximate inverse so constructed. These same three steps will appear below, but to a large extent the 'work' is in the first step — finding and solving the model problems that give an inverse up to compact errors.

I have been, and will be, concentrating on the Hodge decomposition or the generalized inverse of $d + \delta$. This is really a convenient intermediate goal. The same approach gives

THEOREM 3. *On any compact Riemann manifold (without boundary) the closure of $d + \delta$ as an unbounded operator on $L^2(X;\Lambda^*)$ is self-adjoint; its resolvent is a meromorphic family of pseudodifferential operators*

$$(d + \delta - \lambda)^{-1} \in \Psi^{-1}(X;\Lambda^*)$$

with poles at the discrete set of eigenvalues of $d+\delta$ on $\mathbb{R}$ with residues the orthogonal projections onto the corresponding eigenspaces.

The analysis of the resolvent is in turn only a starting point, or a model, for more sophisticated problems and questions. For instance, one might want to look at the heat kernel, the index (signature), the eta invariant and many other objects. My idea is that the methods and results discussed here will make *all* these questions at least approachable in the more general case of stratified varieties.

One corollary of the explicit Hodge decomposition I am discussing here, very much related to the questions below, is Poincaré duality. Since I do/don't have time let me briefly discuss/ignore this.

If we assume that X is *oriented* then the Riemannian density dg lifts to a smooth, positive, sector of $\Lambda^n X$ — I will generally denote this μ_g although dg would probably be more standard. Since $\Lambda^n X$ is one-dimensional the choice of g gives a non-degenerate pairing

$$\Lambda^k_x \times \Lambda^{n-k}_x \ni (\alpha,\gamma) \longmapsto (\alpha,\gamma)/\mu_g \in \mathbb{R}.$$

Since g also fixes a Euclidean inner product on Λ^* this in turn gives an isomorphism, the Hodge $\star$ operator

$$\star : \Lambda^k \longrightarrow \Lambda^{n-k} \quad \forall\ k$$
$$\langle \varphi, \star\psi\rangle \mu_g = \varphi \wedge \psi \quad \forall\ \varphi \in \mathcal{C}^\infty(X,\Lambda^{n-k}),\ \psi \in \mathcal{C}^\infty(X,\Lambda^k)\,.$$

PROPOSITION 5. *The Hodge $\star$ isomorphism induces an isomorphism on Hodge cohomology, $H^*_{\mathrm{Ho}}(X)$, which realizes Poincaré duality.*

This is one thing we wish to recapture for the singular spaces considered below.

The 'model' problem in the interior, i.e. for the problem just discussed, is the inversion of the symbol, $\sigma_1(d+\delta)$. This is intimately related to the corresponding Euclidean problem, which I proceed to discuss. For Euclidean space the dy^α give a global basis for the Λ^k. Since this basis is orthonormal we have

$$\begin{aligned}(d+\delta)\psi &= \sum_{j,\alpha} \frac{\partial \psi_\alpha}{\partial y_j}(dy^j \wedge -\iota_{\partial_j})dy^\alpha \\ &= \sum_{j,\alpha} \frac{\partial \psi_\alpha}{\partial y_j}(\mathrm{ext}(dy^j) - \mathrm{int}(dy^j))\, dy^\alpha \,.\end{aligned}$$

The square of $d+\delta$ is the flat Laplacian

$$(d+\delta)^2\psi = -\sum_{j,\alpha} \frac{\partial^2 \psi_\alpha}{\partial y_j^2}\, dy^\alpha = \Delta\psi \tag{1.2}$$

The flat Laplacian has a 'preferred' fundamental solution:

$$F_n = \begin{cases} \frac{1}{2\pi}\log|y| & n = 2 \\ c_n\,|y|^{2-n} & n \geq 2\,. \end{cases}$$

That is, if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is compactly supported then

$$u(x) = F_n * \psi = \int_{\mathbb{R}^n} F_n(x-y)\psi(y)\, dy$$

satisfies $\Delta u = \psi$. Notice that the integral is well defined since F_n is locally integrable.

It follows that $E_n = (d+\delta)F_n$, where F acts as a convolution operator diagonally on forms, is a fundamental solution for $d+\delta$. Thus E_n is a convolution operator with kernel

$$E_n(y) = c_n' \sum_j \frac{y_j}{|y|^n}(\mathrm{ext}(dy^j) - \mathrm{int}(dy^j))\,. \tag{1.3}$$

This is a *homogeneous* distribution. In fact E_n is by no means unique as a fundamental solution (it is 'preferred' on grounds of homogeneity). The true *model* kernel is the singular part of E_n — since E_n is homogeneous this is the same as E_n itself.

The model kernel in (1.3) determines the main part of the parametrix E' in the sense that it is the leading part of the singularity of E' at each point on the diagonal.

Since it will be needed below, let me recall some of the properties of the eigenforms of $d+\delta$. These fall naturally into two classes. First, the harmonic forms form the null space of $d+\delta$, this is a *graded* space

$$H^*\mathrm{Ho}(X) = \bigoplus_k H^k_{\mathrm{Ho}}(X),\ H^k_{\mathrm{Ho}}(X) = H^*\mathrm{Ho}(X) \cap \mathcal{C}^\infty(X;\Lambda^k X).$$

Notice that the fact that the null space of $d+\delta$ is graded is not completely trivial. Rather if follows from the fact that

$$\begin{aligned}H^*_{\mathrm{Ho}}(X) &= \{u \in \mathcal{C}^\infty(X;\Lambda^* X); (d+\delta)u = 0\} \\ &= \{u \in \mathcal{C}^\infty(X;\Lambda^* X); du = \delta u = 0\}\,.\end{aligned}$$

Indeed, if $(d+\delta)u = 0$ then

$$\begin{aligned} 0 &= ((d+\delta), u(d+\delta)u) \\ &= (du, du) + (du, \delta u) + (\delta u, du) + (\delta u, \delta u) \\ &= (du, du) + (\delta u, \delta u). \end{aligned} \tag{1.4}$$

Thus $du = \delta u = 0$ because $(du, \delta u) = (d^2 u, u) = 0$ by integration by parts. Since d and δ are graded operators it follows that their joint null space is graded. We shall need to carry through this argument in more generality later.

Now, the non-zero[6] eigenforms are *not* graded. Suppose that $(d+\delta)u = \tau u$ with $\tau \neq 0$. We can assume that u is orthogonal to $H^*_{\mathrm{Ho}}(X)$, by the self-adjointness of $d+\delta$. Thus

$$u = u_d + u_\delta$$

is its Hodge decomposition, with u_d exact and u_δ coexact. Applying $d+\delta$ gives

$$du_\delta = \tau u_d, \ \delta u_d = \tau u_\delta.$$

Decomposing $u_d = \sum_k u_d^{(k)}$ and $u_\delta = \sum_k u_\delta^{(k)}$ into their pure form components gives

$$du_\delta^{(k-1)} = \tau u_d^{(k)}, \ \delta u_d^{(k)} = \tau u_\delta^{(k-1)}. \tag{1.5}$$

It follows that there is a basis of eigenforms consisting of the sum of an exact $k-1$ form and a coexact k form satisfying (1.5).

Proposition 6. *The non-zero eigenvalues of $d+\delta$ come in pairs $\pm\tau$ and correspond to pairs of forms satisfying* (1.5) *with $u_\delta^{(k-1)} + u_d^{(k)}$ a τ-eigenform and $u_\delta^{(k-1)} - u_d^{(k)}$ a $(-\tau)$-eigenform.*

[6] By which I mean non-zero eigenforms of non-zero eigenvalues.