Hodge theory of stratified varieties
Lecture notes for 18.157, Spring 1997

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Introduction

In these lectures I plan to describe the Hodge theory for a certain class of spaces which includes stratified varieties; these general spaces are sometimes called *pseudo-manifolds*[1]. Most of the new material here represents joint work with Rafe Mazzeo.

[1] The definition I take is a little wider than the usual one, so as to include manifolds with boundary and even corners.

CHAPTER 1

Hodge theory of compact $\mathcal{C}^\infty$ manifolds

To start[1] with I shall briefly recall the case of a compact manifold without boundary. Since I assume that this standard case is already familiar to you, I shall discuss it here mainly to set the agenda for later.

Let X be a compact manifold with boundary. The tangent bundle TX and its dual T^*X are both well-defined manifolds which have the structure of vector bundles over X. A Riemannian metric on X is a smoothly-varying positive-definite Euclidean product on the fibres of TX. For the most part we assume that such a metric, g, is given.

The exterior powers of T^*X are the form bundles $\Lambda^k X$ with fibres $\Lambda^k_x X = \Lambda^k(T^*_x X)$; their sections are forms on X. The fundamental operator we consider is the de Rham differential. This is a continuous linear operator

$$d : \mathcal{C}^\infty(X;\Lambda^k) \longrightarrow \mathcal{C}^\infty(X;\Lambda^{k+1}) \ \forall\ k.$$

In local coordinates[2] $y_1,\dots,y_n$ in a coordinate neighbourhood O of X, $n=\dim X$, $T^*_x X$ for $x \in O$ has the coordinate basis $dy_1,\dots,dy_n$. This induces a basis of each of the $\Lambda^k_x X$; namely the

$$dy^\alpha = dy_{\alpha_1}\wedge dy_{\alpha_2}\wedge\cdots\wedge dy_{\alpha_k}$$

give a basis of Λ^k_x as α runs over the strictly increasing subsets[3] of $\{1,\dots,n\}$ with k elements. In terms of this basis an element $\phi \in \mathcal{C}^\infty(O,\Lambda^k)$ can be written

$$\phi = \sum_{|\alpha|=k} \phi_\alpha(y)\, dy^\alpha$$

and then

$$d\phi = \sum_{j=1}^n \sum_{|\alpha|=k} \frac{\partial \phi_\alpha}{\partial y_j}\, dy_j \wedge dy^\alpha$$

$$dy_j \wedge dy^\alpha = \begin{cases} 0 & \text{if } j \in \alpha \\ (-1)^{\ell(\alpha,j)}\, dy^\beta & \text{if } j \notin \alpha,\ \beta = \alpha\cup\{j\} \end{cases}$$

with $l(\alpha,j)$ the number of elements of α less than j.

The fundamental property of d is that $d^2=0$ so we get the de Rham complex

$$\mathcal{C}^\infty(X) \xrightarrow{d} \mathcal{C}^\infty(X;T^*X) \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^2 X) \xrightarrow{d} \cdots$$
$$\cdots \xrightarrow{d} \mathcal{C}^\infty(X;\Lambda^n) \xrightarrow{d} 0.$$

[1] Thanks To Victor Nistor for helpful comments on this chapter.

[2] Thus there is a diffeomorphism $O \overset{F}{\longleftrightarrow} O' \subset \mathbb{R}^n$ onto an open set such that the y_j on the pull-backs under F of the standard coordinates on $\mathbb{R}^n$.

[3] Of course I am really taking α to be an increasing *map* $\alpha : \{1,\dots,k\} \longrightarrow \{1,\dots,n\}$.

The homology of this complex is the ($\mathcal{C}^\infty$) de Rham cohomology of X

$$H^k_{\mathrm{dR}}(X) = \frac{\{u \in \mathcal{C}^\infty(X;\Lambda^k); du = 0\}}{d\mathcal{C}^\infty(X;\Lambda^{k-1})}.$$

THEOREM 1. *(De Rham) For a compact $\mathcal{C}^\infty$ manifold $H^k_{\mathrm{dR}}(X)$ has a natural non-degenerate pairing with $H_{k,\Delta}(X;\mathbb{R})$ the simplicial (or the singular, or the Čech ...) homology with real coefficients:*

$$\langle u, \sigma\rangle = \sum_i c_i \int_{\Delta_k} m^*_{\sigma_1} u,\ \ \sigma = \sum_i c_i m_i\ .$$

The space of square-integrable sections of a vector bundle on X is also well-defined by reference to local coordinates. Then d can be extended to a densely-defined operator on L^2 by setting

$$\mathrm{Dom}^k(d) = \{u \in L^2(X;\Lambda^k);\ du \in L^2(X;\Lambda^{k+1})\}$$

where the second condition can be taken in either a strong or a weak sense. Given such a definition, the L^2 cohomology of X is

$$H^k_{L^2}(X) = \frac{\left\{u \in \mathrm{Dom}^k(X); du = 0\right\}}{d\,\mathrm{Dom}^{k-1}(X)} \cong H^k_{\mathrm{dR}}(X),$$

another form of the de Rham theorem.

The main thrust of these lectures is about Hodge theory. This uses the choice of a metric g. By definition this is a Euclidean inner product on the fibres of TX; by duality it fixes a Euclidean inner product in the fibres of T^*X. Let $e_1, \dots, e_n$ be an orthonormal basis of T^*_xX. The assertion that the $e^\alpha = e_{\alpha_1} \wedge e_{\alpha_2} \wedge \dots \wedge e_{\alpha_k}$ form an orthonormal basis of Λ^k_x fixes a Euclidean inner product on Λ^k_xX. It is independent of the choice of orthonormal basis of T^*_xX. Thus the Λ^kX also have, smoothly varying, Euclidean inner products on their fibres. I shall denote these inner products by $\langle\bullet,\bullet\rangle$. The density bundle on X also has a natural smooth section fixed by the metric; in terms of any orthonormal basis its value is

$$|e_1 \wedge \cdots \wedge e_n| \in \Omega_n X.$$

I shall denote this Riemannian density dg.

The fibre inner product fixes a global inner product on $\mathcal{C}^\infty(X;\Lambda^k)$:

$$(\phi,\phi') = \int_X \langle\phi,\phi'\rangle\, dg_x\ .$$

This in turn determines a second differential operator

$$\delta : \mathcal{C}^\infty(X;\Lambda^k) \to \mathcal{C}^\infty(X;\Lambda^{k-1})$$

for each k, by

$$(du, v) = (u, \delta v)\ \forall\ u \in \mathcal{C}^\infty(X;\Lambda^{k-1}),\ v \in \mathcal{C}^\infty(X;\Lambda^k).$$

THEOREM 2. *(Hodge decomposition) The operator $d+\delta$ is Fredholm and self-adjoint on $\mathcal{C}^\infty(X;\Lambda^*)$; if $E : \mathcal{C}^\infty(X;\Lambda^*) \longrightarrow \mathcal{C}^\infty(X;\Lambda^*)$ is the unique generalized inverse which has range in the orthocomplement and annihilates the null space, $H^*_{\mathrm{Ho}}(X)$, of $d+\delta$ then*

$$\mathcal{C}^\infty(X;\Lambda^k) = dE\mathcal{C}^\infty(X;\Lambda^k) \oplus \delta E\mathcal{C}^\infty(X;\Lambda^k) \oplus H^k_{\mathrm{Ho}}(X)$$

is the decomposition into exact, coexact and harmonic k-forms.

Corollary 1. $H^k_{\mathrm{Ho}}(X) \cong H^k_{\mathrm{dR}}(X)$.

It is this form of the 'Hodge theorem' that I wish to generalize to various 'singular' situations. Let me briefly review the proof since this should serve to motivate some of the later enterprises. The *proof* can be reduced to three steps leading to the construction of E.

1. Construct a 'crude' parametrix E' for $d+\delta$, such that $(d+\delta)\circ E' - \mathrm{Id}$ and $E'\circ(d+\delta) - \mathrm{Id}$ are both compact operators on $L^2(X;\Lambda^* X)$.
2. Refine the parametrix to get an operator E'' such that $(d+\delta)\circ E'' - \mathrm{Id}$, $E''\circ(d+\delta) - \mathrm{Id}$ are both smoothing operators on $\mathcal{C}^\infty(X;\Lambda^* X)$.
3. Conclude that $d+\delta$ is Fredholm on $\mathcal{C}^\infty(X;\Lambda^* X)$ and that $E - E''$ is a smoothing operator.

To carry out these three steps it is very convenient to use the algebra of pseudo-differential operators $\Psi^{\mathbb{Z}}(X;\Lambda^* X)$. This consists of operators on $\mathcal{C}^\infty(X;\Lambda^* X)$. The filtration by order has a simple corresponding graded quotient i.e. there is a short exact sequence for each $k\in\mathbb{Z}$

$$
\begin{aligned}
(1.1)\quad 0 \longrightarrow \Psi^{k-1}(X;\Lambda^* X) \longrightarrow \Psi^{k}(X;\Lambda^* X) &\xrightarrow{\sigma_k} \\
&\xrightarrow{\sigma_k} \mathcal{C}^\infty(S^*X; L^k\otimes \mathrm{Hom}(\Lambda^* X)) \to 0.
\end{aligned}
$$

Here L is the (trivial) line bundle over $S^*X = (T^*X\setminus 0)/\mathbb{R}^+$ corresponding to functions on $T^*X\setminus 0$ which are homogeneous of degree 1.[4] This sequence is multiplicative

$$\sigma_{k+l}(AB) = \sigma_k(A)\cdot\sigma_l(B),\ \forall\ A\in\Psi^k(X;\Lambda^* X),\ B\in\Psi^l(X;\Lambda^* X).$$

Furthermore, differential operators *are* pseudo-differential operators:

$$d+\delta\in\mathrm{Diff}^1(X;\Lambda^* X)\subset\Psi^1(X;\Lambda^* X).$$

The *symbol* of a differential operator is easily computed and in this case

$$
\begin{aligned}
\sigma_1(d) = i\xi\wedge = i\,\mathrm{ext}(\xi) \in \mathcal{C}^\infty(S^*X; L\otimes\mathrm{hom}(\Lambda^* X))\\
\sigma_1(\delta) = (i\xi\lambda)^* = -i\,\mathrm{int}(\xi)\in\mathcal{C}^\infty(S^*X; L\otimes\mathrm{hom}(\Lambda^* X)).
\end{aligned}
$$

Here $\mathrm{int}(\xi)$, for $\xi\in T^*X$ is the action by inner product.[5]

Now, it follows from this that $\sigma_1(d+\delta)$ is an *invertible* element of $\mathcal{C}^\infty(S^*X; L\otimes \mathrm{hom}(\Lambda^*))$, with inverse in $\mathcal{C}^\infty(S^*X; L^{-1}\otimes\mathrm{hom}(\Lambda^*))$. The exactness of the sequence (1.1) means there exists

$$E'\in\Psi^{-1}(X;\Lambda^*)\text{ with }\sigma_{-1}(E') = \sigma_1(d+\delta))^{-1}.$$

This operator has the desired properties if we observe that

Proposition 1. *Elements of the space $\Psi^{-1}(X;\Lambda^* X)$ define compact operators on $L^2(X;\Lambda^* X)$.*

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[4] The dual metric defines (or is) a function which is positive and homogeneous of degree 2 on $T^*X\setminus 0$. This gives a trivialization of L.

[5] This is determined by the metric and contraction. If e_i is an orthonormal basis of $T^*_x X$ with $e_1 = \xi/|\xi|$ then

$$\mathrm{int}(\xi)e^\alpha = \begin{cases} 0 & \alpha_1\neq 0\\ |\xi|e^{\alpha'}, & \alpha = (\alpha_1,\alpha').\end{cases}$$

It is independent of choice of basis.

To proceed to the second step of the construction consider the 'remainder'

$$(d+\delta)\circ E' = \operatorname{Id} - R_1, \quad R_1 \in \Psi^{-1}(X;\Lambda^* X).$$

The formal Neumann series for the inverse of the right side

$$\operatorname{Id} + R_1 + R_1^2 + \cdots + R_1^k + \cdots$$

has kth term in $\Psi^{-k}(X;\Lambda^*)$. Whilst it will not converge in general we can apply the following form of 'Borel's lemma' to asymptotically sum the series.

Proposition 2. *(Asymptotic completeness) For any series $\sum_j A_j$ with terms $A_j \in \Psi^{-j}(X;\Lambda^*)$ there exists $A \in \Psi^{\mathbb{Z}}(X;\Lambda^*)$ such that $A - \sum_{j<k} A_j \in \Psi^{-k}(X;\Lambda^*)$ for* all k; *furthermore A is unique modulo a term in*

$$\Psi^{-\infty}(X;\Lambda^*) = \bigcap_k \Psi^k(X;\Lambda^*).$$

The 'residual terms' here are easily understood.

Proposition 3. *The two-sided ideal $\Psi^{-\infty}(X;\Lambda^*) \subset \Psi^{\mathbb{Z}}(X;\Lambda^*)$ consists precisely of the smoothing operators, meaning those operators on $\mathcal{C}^\infty(X;\Lambda^*)$ which can be written as integral operators with kernel $A \in \mathcal{C}^\infty(X^2;\hom(\Lambda^*)\otimes\Omega_{\mathbb{R}})$*

$$Au(x) = \int_X A(x,y)u(y)\,.$$

Applying the proposition on asymptotic completeness gives an element $S \in \Psi^{-1}(X;\Lambda^*)$ such that

$$(\operatorname{Id} - R_1)\circ(\operatorname{Id} + S) = \operatorname{Id} + R_2, R_2 \in \Psi^{-\infty}(X;\Lambda^*)\,.$$

Then $E'' = E'\circ(\operatorname{Id} - S)$ is an operator with the properties stated. That $(d+\delta)\cdot E'' - \operatorname{Id}$ is a smoothing operator is immediate by construction. Similarly $(\operatorname{Id} - S)\circ E''$ is a *left parametrix*, i.e. an inverse in $\Psi^{\mathbb{Z}}(X;\Lambda^*)/\Psi^{-\infty}(X;\Lambda^*)$. Since this is a ring, left and right inverses coincide so E'' is *also* a left parametrix.

At this point it has been established that $d+\delta : \mathcal{C}^\infty(X;\Lambda^*) \longrightarrow \mathcal{C}^\infty(X;\Lambda^*)$ is Fredholm, i.e. has a finite dimensional null space, denoted $H^*_{\mathrm{Ho}}(X)$, and range of finite codimension (hence closed). The self-adjointness means that $H^*_{\mathrm{Ho}}(X)$ must be a complement to the range. Hence there is a unique operator E which has null space $H^*_{\mathrm{Ho}}(X)$, range $H^*_{\mathrm{Ho}}(X)^\perp$ and inverts $d+\delta$ on its range.

It remains only to show that $E - E'' \in \Psi^{-\infty}(X;\Lambda^*)$, and hence to see that $E \in \Psi^{-1}(X;\Lambda^*)$. This follows from the 'bi-ideal' property of $\Psi^{-\infty}(X;\Lambda^*)$.

Proposition 4. *If $A : \mathcal{C}^\infty(X;\Lambda^*) \longrightarrow \mathcal{C}^{-\infty}(X;\Lambda^*)$ (distributional sections) is any continuous linear operator and $P, Q \in \Psi^{-\infty}(X;\Lambda^*)$ then $PAQ \in \Psi^{-\infty}(X;\Lambda^*)$.*

Thus E and E' satisfy

$$PE - \operatorname{Id} = Q,\ PE' - \operatorname{Id} = Q' \text{ and}$$
$$EP - \operatorname{Id} = R,\ E'P - \operatorname{Id} = R'$$

with $Q, Q', R, R' \in \Psi^{-\infty}(X;\Lambda^* X)$. Combining the first and last of these equations gives

$$E - E' = E'Q - R'E.$$

Similarly combining the other two gives

$$E - E' = E'R - EQ'.$$

Inserting the second of these in the first gives

$$E - E' = E'Q - R'E' - R'E'Q + R'EQ'.$$

All terms on the right are in $\Psi^{-\infty}(X;\Lambda^*X)$, the last by Proposition 4.

Now, my objective below is to give a similar Hodge decomposition, and a similar proof of it, for certain singular spaces. With this in mind let us observe that the three steps in the proof — which is the construction of E — have the following flavour. First, we solve a *model problem.* In this case it is the inversion of $\sigma_1(d+\delta)$; this is closely related to the Hodge theory of Euclidean space. Secondly an iterative argument allows the 'error' to be refined so that we can discuss rather precisely the mapping properties of $d+\delta$. The third (and somewhat optional) step is to show that the 'true inverse' E is actually very close to the approximate inverse so constructed. These same three steps will appear below, but to a large extent the 'work' is in the first step — finding and solving the model problems that give an inverse up to compact errors.

I have been, and will be, concentrating on the Hodge decomposition or the generalized inverse of $d+\delta$. This is really a convenient intermediate goal. The same approach gives

THEOREM 3. *On any compact Riemann manifold (without boundary) the closure of $d+\delta$ as an unbounded operator on $L^2(X;\Lambda^*)$ is self-adjoint; its resolvent is a meromorphic family of pseudodifferential operators*

$$(d+\delta-\lambda)^{-1} \in \Psi^{-1}(X;\Lambda^*)$$

with poles at the discrete set of eigenvalues of $d+\delta$ on $\mathbb{R}$ with residues the orthogonal projections onto the corresponding eigenspaces.

The analysis of the resolvent is in turn only a starting point, or a model, for more sophisticated problems and questions. For instance, one might want to look at the heat kernel, the index (signature), the eta invariant and many other objects. My idea is that the methods and results discussed here will make *all* these questions at least approachable in the more general case of stratified varieties.

One corollary of the explicit Hodge decomposition I am discussing here, very much related to the questions below, is Poincaré duality. Since I do/don't have time let me briefly discuss/ignore this.

If we assume that X is *oriented* then the Riemannian density dg lifts to a smooth, positive, sector of $\Lambda^n X$ — I will generally denote this μ_g although dg would probably be more standard. Since $\Lambda^n X$ is one-dimensional the choice of g gives a non-degenerate pairing

$$\Lambda^k_x \times \Lambda^{n-k}_x \ni (\alpha,\gamma) \longmapsto (\alpha,\gamma)/\mu_g \in \mathbb{R}\,.$$

Since g also fixes a Euclidean inner product on Λ^* this in turn gives an isomorphism, the Hodge $\star$ operator

$$\star : \Lambda^k \longrightarrow \Lambda^{n-k} \quad \forall\ k$$

$$\langle\varphi,\star\psi\rangle\mu_g = \varphi\wedge\psi \quad \forall\ \varphi\in\mathcal{C}^\infty(X,\Lambda^{n-k}),\ \psi\in\mathcal{C}^\infty(X,\Lambda^k)\,.$$

PROPOSITION 5. *The Hodge $\star$ isomorphism induces an isomorphism on Hodge cohomology, $H^*_{\mathrm{Ho}}(X)$, which realizes Poincaré duality.*

This is one thing we wish to recapture for the singular spaces considered below.

The 'model' problem in the interior, i.e. for the problem just discussed, is the inversion of the symbol, $\sigma_1(d+\delta)$. This is intimately related to the corresponding Euclidean problem, which I proceed to discuss. For Euclidean space the dy^α give a global basis for the Λ^k. Since this basis is orthonormal we have

$$\begin{aligned}(d+\delta)\psi &= \sum_{j,\alpha} \frac{\partial \psi_\alpha}{\partial y_j}(dy^j \wedge -\iota_{\partial_j})dy^\alpha \\ &= \sum_{j,\alpha} \frac{\partial \psi_\alpha}{\partial y_j}(\operatorname{ext}(dy^j) - \operatorname{int}(dy^j))\, dy^\alpha\,.\end{aligned}$$

The square of $d+\delta$ is the flat Laplacian

$$(d+\delta)^2\psi = -\sum_{j,\alpha} \frac{\partial^2 \psi_\alpha}{\partial y_j^2}\, dy^\alpha = \Delta\psi \tag{1.2}$$

The flat Laplacian has a 'preferred' fundamental solution:

$$F_n = \begin{cases} \frac{1}{2\pi}\log|y| & n=2 \\ c_n\,|y|^{2-n} & n \geq 2\,. \end{cases}$$

That is, if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is compactly supported then

$$u(x) = F_n * \psi = \int_{\mathbb{R}^n} F_n(x-y)\psi(y)\,dy$$

satisfies $\Delta u = \psi$. Notice that the integral is well defined since F_n is locally integrable.

It follows that $E_n = (d+\delta)F_n$, where F acts as a convolution operator diagonally on forms, is a fundamental solution for $d+\delta$. Thus E_n is a convolution operator with kernel

$$E_n(y) = c_n' \sum_j \frac{y_j}{|y|^n}(\operatorname{ext}(dy^j) - \operatorname{int}(dy^j))\,. \tag{1.3}$$

This is a *homogeneous* distribution. In fact E_n is by no means unique as a fundamental solution (it is 'preferred' on grounds of homogeneity). The true *model* kernel is the singular part of E_n — since E_n is homogeneous this is the same as E_n itself.

The model kernel in (1.3) determines the main part of the parametrix E' in the sense that it is the leading part of the singularity of E' at each point on the diagonal.

Since it will be needed below, let me recall some of the properties of the eigenforms of $d+\delta$. These fall naturally into two classes. First, the harmonic forms form the null space of $d+\delta$, this is a *graded* space

$$H^*\mathrm{Ho}(X) = \bigoplus_k H^k_{\mathrm{Ho}}(X),\ H^k_{\mathrm{Ho}}(X) = H^*\mathrm{Ho}(X) \cap \mathcal{C}^\infty(X;\Lambda^k X).$$

Notice that the fact that the null space of $d+\delta$ is graded is not completely trivial. Rather if follows from the fact that

$$\begin{aligned}H^*_{\mathrm{Ho}}(X) &= \{u \in \mathcal{C}^\infty(X;\Lambda^* X); (d+\delta)u = 0\} \\ &= \{u \in \mathcal{C}^\infty(X;\Lambda^* X); du = \delta u = 0\}\,.\end{aligned}$$

Indeed, if $(d+\delta)u = 0$ then

$$\begin{aligned} 0 &= ((d+\delta), u(d+\delta)u) \\ &= (du, du) + (du, \delta u) + (\delta u, du) + (\delta u, \delta u) \\ &= (du, du) + (\delta u, \delta u). \end{aligned} \tag{1.4}$$

Thus $du = \delta u = 0$ because $(du, \delta u) = (d^2 u, u) = 0$ by integration by parts. Since d and δ are graded operators it follows that their joint null space is graded. We shall need to carry through this argument in more generality later.

Now, the non-zero[6] eigenforms are *not* graded. Suppose that $(d+\delta)u = \tau u$ with $\tau \neq 0$. We can assume that u is orthogonal to $H^*_{\mathrm{Ho}}(X)$, by the self-adjointness of $d+\delta$. Thus

$$u = u_d + u_\delta$$

is its Hodge decomposition, with u_d exact and u_δ coexact. Applying $d+\delta$ gives

$$du_\delta = \tau u_d, \ \delta u_d = \tau u_\delta.$$

Decomposing $u_d = \sum_k u_d^{(k)}$ and $u_\delta = \sum_k u_\delta^{(k)}$ into their pure form components gives

$$du_\delta^{(k-1)} = \tau u_d^{(k)}, \ \delta u_d^{(k)} = \tau u_\delta^{(k-1)}. \tag{1.5}$$

It follows that there is a basis of eigenforms consisting of the sum of an exact $k-1$ form and a coexact k form satisfying (1.5).

PROPOSITION 6. *The non-zero eigenvalues of $d+\delta$ come in pairs $\pm\tau$ and correspond to pairs of forms satisfying* (1.5) *with $u_\delta^{(k-1)} + u_d^{(k)}$ a τ-eigenform and $u_\delta^{(k-1)} - u_d^{(k)}$ a $(-\tau)$-eigenform.*

[6] By which I mean non-zero eigenforms of non-zero eigenvalues.

CHAPTER 2

Compact manifolds with boundary

1. Boundaries

As the first 'singular' case of Hodge theory I will consider a compact manifold with boundary, X. All the usual analytic objects on a manifold without boundary extend to such a space. Thus the total spaces of the bundles TX, T^*X, $\Lambda^k X$ are themselves manifolds with boundary. One way to finesse this discussion is to *double* X.

PROPOSITION 7. *If X is a compact manifold with boundary the space*

$$2X = (X \sqcup X)/i, \ i : \partial X \sqcup \emptyset \longrightarrow \emptyset \sqcup \partial X$$

can be given the structure of a compact manifold without boundary so that the reflection map is a diffeomorphism and

$$\mathcal{C}^\infty(X) = \mathcal{C}^\infty(2X)\big|_X.$$

Then all the objects such as $C^\infty(X; \Lambda^*)$ are just the restrictions[1] *of the corresponding objects to X.*

In any case I take it as established that the de Rham differential and complex continue to make sense

$$\mathcal{C}^\infty(X) \xrightarrow{d} \mathcal{C}^\infty(X; T^*X) \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^2 X) \xrightarrow{d} \cdots$$
$$\cdots \xrightarrow{d} \mathcal{C}^\infty(X; \Lambda^n X) \longrightarrow 0.$$

The homology of this complex is called the (absolute) de Rham cohomology of X.

$$H^k_{dR}(X) = H^k_{dR,A}(X) = \frac{\{u \in \mathcal{C}^\infty(X; \Lambda^k X); du = 0\}}{d\mathcal{C}^\infty(X; \Lambda^{k-1} X)}. \tag{2.1}$$

The de Rham theorem extends to say that this is isomorphic to your favorite cohomology of X (with coefficients $\mathbb{R}$) i.e. simplicial, singular or Čech. Note that the *duality* statement is that $H^k_{dR,A}(X)$ pairs naturally with the *relative* homology, not the absolute homology.

Since the boundary of X is a 'natural' submanifold we can look at the objects which vanish there to all orders. Thus

$$\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(X) \tag{2.2}$$

is, by definition, the subspace of smooth functions vanishing with their Taylor series at ∂X. This can also be identified as the *subspace* of $\mathcal{C}^\infty(2X)$ consisting of those functions which vanish identically on the other X component, i.e. can

[1] In fact it still needs to be shown that these restrictions are independent of choices involved in producing $2X$.

be extended as zero across the boundary. This same characterization allows us to define $\dot{\mathcal{C}}^\infty(X;\Lambda^k X)$ etc. In fact we get a subcomplex of the de Rham complex for X (or for $2X$):

$$\dot{\mathcal{C}}^\infty(X) \xrightarrow{d} \dot{\mathcal{C}}^\infty(X;T^*X) \xrightarrow{d} \dot{\mathcal{C}}^\infty(X;\Lambda^2 X) \xrightarrow{d} \cdots$$
$$\cdots \xrightarrow{d} \dot{\mathcal{C}}^\infty(X;\Lambda^n X) \xrightarrow{d} 0.$$

The homology of this complex is the *relative* de Rham cohomology of X (i.e. relative to the boundary)

$$H^k_{dR,R}(X) = H^k_{dR}(X,\partial X) = \frac{\left\{u \in \dot{\mathcal{C}}^\infty(X;\Lambda^k X); du = 0\right\}}{d\dot{\mathcal{C}}^\infty(X;\Lambda^{k-1}X)}.$$

Again the de Rham theorem extends to show that this is isomorphic to the relative cohomology of X in other senses, e.g. simplicial or Čech; it pairs in a non-degenerate fashion with the absolute homology.

2. Metrics

Next consider the notion of a metric on X. As we shall see later, there really are several distinct choices about what this should mean. The *obvious* one is the restriction of a metric from $2X$ and this just means a fibre metric on TX which is 'smooth up to the boundary'. Any such metric takes a relatively simple form in the 'correct' coordinates.

PROPOSITION 8. *If X is a (compact) manifold with boundary and g is a metric on X then there is a neighborhood O of ∂X in X and a diffeomorphism*

$$O \xrightarrow{F} [0,\epsilon)_x \times \partial X$$

such that $g = F^*(g')$ when g' is almost a product metric

$$g' = dx^2 + h(x), \tag{2.3}$$

h being a metric on ∂X depending smoothly on $x \in [0,\epsilon)$.

PROOF. In fact x is the distance to the boundary for the metric g and F is made up by flow along the geodesics normal to the boundary. □

For various computations below it is convenient to work with a metric which is genuinely a product near the boundary,

$$g = dx^2 + h. \tag{2.4}$$

In fact nothing I shall do really depends on this and in most instances (except where there are explicit indications to the contrary) the general case is little more than notationally harder.

If we work with the spaces $\dot{\mathcal{C}}^\infty(X;\Lambda^*X)$ then no boundary terms will arise from integration by parts. This allows us to define the differential operator δ as the *formal* adjoint of d. Thus

$$(du,v) = \int_X \langle du, v\rangle dg = \int_X \langle u, \delta v\rangle dg = (u,\delta v)$$
$$\forall\ u \in \dot{\mathcal{C}}^\infty(X;\Lambda^*X),\ \ v \in \mathcal{C}^\infty(X;\Lambda^*X)$$

does fix $\delta : \mathcal{C}^\infty(X;\Lambda^k X) \longrightarrow \mathcal{C}^\infty(X;\Lambda^{k-1}X)$ for every k. It is the same operator as we get by restriction from $2X$.

We need to consider the true adjoint formula. For this it is useful to get a rather explicit representation of d and δ in terms of a product decomposition as in Proposition 8.

From (2.4) (and even (2.3)) the 1-form dx has length 1, near ∂X. Any form can be resolved into its tangential and normal components

$$u = \phi + dx \wedge \psi \quad \text{on } O$$

where $\phi \in \mathcal{C}^\infty(O;\Lambda^k(\partial X))$ and $\psi \in \mathcal{C}^\infty(O;\Lambda^{k-1}(\partial X))$ are forms on ∂X depending on x as a parameter. For the metric this decomposition of the form bundles

$$\Lambda^k X = \Lambda^k \partial X \oplus \Lambda^{k-1}\partial X \tag{2.5}$$

is orthogonal, with the inner product on ∂X being that defined by h.

In terms of this decomposition

$$du = d_t\phi + dx \wedge \frac{\partial\phi}{\partial x} - dx \wedge d_t\psi$$

where d_t is the de Rham differential on ∂X.[2] Written out as a 2×2 matrix in terms of (2.5)

$$d = \begin{pmatrix} d & 0 \\ \partial_x & -d \end{pmatrix}. \tag{2.6}$$

The orthogonality of the decomposition means that if v similarly has tangential part ϕ' and normal part ψ' then

$$\langle u, v\rangle_g = \langle \phi, \phi'\rangle_h + \langle \psi, \psi'\rangle_h \quad \text{near } \partial X.$$

Inserting this into the adjoint formula,

$$\begin{aligned}\int \langle du, v\rangle_g dg \\ &= \int_0^\delta \int_{\partial X} \left(\langle d\phi, \phi'\rangle_h + \langle \partial_x\phi, \psi'\rangle_h - \langle d\psi, \psi'\rangle_h\right) dh dx \\ &\qquad = \int_0^\delta \int_{\partial X} \left(\langle \phi, \delta\phi'\rangle_h - \langle \phi, \partial_x\psi'\rangle_h - \langle \psi, \delta\psi'\rangle_h\right) dh dx\end{aligned}$$

where I have assumed that $u \in \dot{\mathcal{C}}^\infty(X;\Lambda^* X)$ has support in $x < \epsilon$. Thus there are again no boundary terms and

$$\delta = \begin{pmatrix} \delta & -\partial_x \\ 0 & -\delta \end{pmatrix} \text{ near } \partial X, \tag{2.7}$$

where on the right δ is the adjoint of d for the metric h.

Proposition 9. *(Green's Theorem) If $u,\ v \in \mathcal{C}^\infty(X, \Lambda^* X)$*

$$b(u, v) = (du, v) - (u, \delta v) = -(Ru, Av) \tag{2.8}$$

and hence

$$\begin{aligned} G(u, v) &= ((d+\delta)u, v) - (u, (d+\delta)v) \\ &= (Au, Rv) - (Ru, Av) \end{aligned} \tag{2.9}$$

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[2] I will drop the subscript 't' and rely on context to distinguish between the tangent and complete operators d and δ.

where the pairing on the right is on ∂X and is with respect to the metric $h(0)$ and $Ru = \phi(o)$ and $Au = \psi(0)$ are the restrictions to ∂X of the tangential and normal parts of u.

Notice that $Ru = i^*_{\partial X} u$, $i_{\partial X} : \partial X \hookrightarrow X$ being inclusion. Similarly

$$Au = i^*_{\partial X}(\iota_\nu u) = i^*_{\partial X}(\mathrm{int}(dx)u),$$

where $\nu = \partial_x$ is the unit inward normal to the boundary.

Formula (2.9) leads rather naturally to the two standard *boundary conditions* for $d + \delta$. Namely they are

$$Ru = i^*_{\partial X} u = 0 \text{ (Relative)} \tag{2.10}$$

and

$$Au = i^*_{\partial X}(\iota_\nu u) = 0 \text{ (Absolute)}. \tag{2.11}$$

Thus we get

$$\begin{aligned} \mathrm{Dom}^k_R(X) &= \left\{u \in \mathcal{C}^\infty(X; \Lambda^k X); Ru = i^*_{\partial X} u = 0\right\} \\ \mathrm{Dom}^k_A(X) &= \left\{u \in \mathcal{C}^\infty(X; \Lambda^k X); Au = i^*_{\partial X}(\iota_\nu u) = 0\right\}. \end{aligned}$$

The first boundary condition does not depend on the metric whereas the second does.

Notice from (2.6), or the universal fact that d commutes with pull-back, that

$$dR = Rd \tag{2.12}$$

and similarly from (2.7) that

$$A\delta = -\delta A \tag{2.13}$$

relates the interior and boundary codifferentials.

3. Relative and absolute decompositions

Essentially the same results hold for $d + \delta$ with these domains as in the boundaryless case.

THEOREM 4. *(Hodge) For $B = R$ or A, $d + \delta$ is self-adjoint*[3] *and Fredholm as an operator*

$$d + \delta : \mathrm{Dom}^*_B(X) \longrightarrow \mathcal{C}^\infty(X; \Lambda^*)$$

with null space, denoted $H^*_{\mathrm{Ho},B}(X)$, also a complement to the range. If E_B is the corresponding generalized inverse then

$$\mathcal{C}^\infty(X; \Lambda^k) = dE_B\mathcal{C}^\infty(X; \Lambda^k) \oplus \delta E_B(X; \Lambda^k) \oplus H^k_{\mathrm{Ho},B}(X)$$

and there is a natural isomorphism

$$H_{dR,B}(X) \cong H^k_{\mathrm{Ho},B}(X), \quad B = A, R.$$

[3] Self-adjointness here and below does not generally to be taken in the Hilbert space sense. Here it means that

$$\begin{aligned} \{u \in \mathcal{C}^\infty(X; \Lambda^* X);\ \exists\ w \in \mathcal{C}^\infty(X; \Lambda^* X) \text{ with } (u, (d+\delta)v) = (w, v) \\ \forall\ v \in \mathrm{Dom}^*_B(X)\} = \mathrm{Dom}^*_B(X). \end{aligned}$$

This follows from (2.9).

Again I shall not discuss the details of the proof of this theorem, *in general,* until later. However the main components of the proof are the same as in the boundaryless case. That is, we analyze model problems and then incorporate these models into a parametrix, which with the help of an algebra of operators can be used to obtain a detailed description of E_B.

In fact there are a number of variants of this approach. One is to 'reduce to the boundary'. This is really quite an old idea called the 'method of double layer potentials' or in a more modern guise is the method of Calderón. It involves the use of the algebra of pseudodifferential operators on the boundary, together with algebra of pseudodifferential operators on the double, $2X$. A second closely related method is to use the algebra of 'pseudodifferential operators satisfying the transmission condition'. This was described by Boutet de Monvel ([**1**]) and developed further by Grubb ([**3**], [**2**]).

I shall use a slightly larger algebra than that introduced by Boutet de Monvel. This was introduced in [**4**] and [**5**]. For the moment I shall simply hypothesize the existence of this algebra and use its formal properties to describe E_B. Justification will come (much) later.

In fact, for this particular boundary problem there is a simple method which can be used when the metric is a product near the boundary. This is the 'method of reflection' which I shall describe below.

The main part of the first step in the 'general method' for constructing E_A or E_R is the analysis of corresponding model problems. For an elliptic boundary problem such as this there are two distinct models. First there is the interior model, in this case the Euclidean problem, just as in the case of a manifold without boundary. Secondly there is a new model. In this case it is the boundary problem for a Euclidean half-space.

Consider $\mathbb{R}^{n,1} = [0,\infty) \times \mathbb{R}^{n-1}$. The first variable will be denoted x and the remaining variables y_j where $j = 1, \cdots, n-1$. The model problem we must solve is

$$(d+\delta)E_B(x,y) = \delta(x-1)\delta(y)\,\mathrm{Id} \tag{2.14}$$

$$BE(0,y) = 0 \tag{2.15}$$

Here B is the boundary operator, $B = A$ or R.

To construct E_B we first consider boundary problems for the Laplacian. Namely in the same half-space $\mathbb{R}^{n,1}$ we wish to find explicit fundamental solutions for the Dirichlet and Neumann problems.

$$\begin{cases} \Delta F_{D,n}(x,y) = \delta(x-1)\delta(y) & x > 0 \\ F_{D,n}(0,y) = 0 \end{cases}$$

and

$$\begin{cases} \Delta F_{N,n}(x,y) = \delta(x-1)\delta(y) & x > 0 \\ \partial_x F_{N,n}(0,y) = 0 \end{cases}$$

In both cases this can be easily accomplished by the method of reflection.

The preferred fundamental solution (1.3), shifted in the first variable, can be used to remove the Dirac delta. Since this function $F_n(x-1,y)$ is singular only at $x = 1$ reflection adds a term which is smooth in $x \geq 0$:

$$\begin{aligned} F_{D,n}(x,y) &= F_n(x-1,y) - F_n(-x-1,y) \\ F_{N,n}(x,y) &= F_n(x-1,y) + F_n(-x-1,y). \end{aligned}$$

The boundary conditions hold in view of the antisymmetry and symmetry in the first variable.

The desired fundamental solutions to $d+\delta$ can now be obtained by applying $d+\delta$ to the appropriate combinations of Dirichlet and Neumann kernels:

$$\begin{aligned} E_A(x,y) &= (d+\delta)\left(F_{D,n}(x,y)\Pi_t + F_{N,n}(x,y)\Pi_n\right) \\ E_R(x,y) &= (d+\delta)\left(F_{N,n}(x,y)\Pi_t + F_{D,n}(x,y)\Pi_n\right). \end{aligned}$$

Here, in terms of the tangential-normal decomposition of forms $\phi = \phi_t + dx\wedge\phi_n$, Π_t is projection into the first term and Π_n is projection onto the second. Thus E_B, for $B = A, R$, is a distribution (in fact a locally integrable function) with values in the homomorphisms of Λ^*. Using the notation

$$\mathrm{cl}(\xi) = i(\mathrm{ext}(\xi) - \mathrm{int}(\xi))$$

for the Clifford action of $\xi\in\mathbb{R}^n$ we can obtain an explicit formula for E_A as a matrix of operators on $\Lambda^*\mathbb{R}^{n-1}$:

$$E_A(x,y) = c_n'\begin{pmatrix} E_{11}\mathrm{cl}(y\cdot dy) & E_{12} \\ E_{21} & E_{22}\mathrm{cl}(y\cdot dy)\end{pmatrix}$$

where

$$\begin{aligned} E_{11} &= \frac{1}{i}\left(|(x-1,y)|^{-n} - |(x+1,y)|^{-n}\right) \\ E_{12} &= -(x-1)\,|(x-1,y)|^{-n} - (x+1)\,|(x+1,y)|^{-n} \\ E_{21} &= (x-1)\,|(x-1,y)|^{-n} - (x+1)\,|(x+1,y)|^{-n} \quad\text{and} \\ E_{22} &= -\frac{1}{i}\left(|(x-1,y)|^{-n} + |(x+1,y)|^{-n}\right). \end{aligned}$$

A similar formula can be obtained for E_R.

As opposed to the interior model, which is homogeneous in all variables, the *global* properties of these model kernels are important. Before describing these, notice that $E_B(x,y)$ is only singular at $x=1, y=0$ and the singularity there is exactly the same as for the interior problem; E_B is smooth up to $x=0$. This fact reappears later as the compatibility between interior and boundary models.

To discuss the global properties of E_B we consider the radial compactification of the half-space to a quarter-space (or a half-ball diffeomorphically)

$$(x,y)\longmapsto \frac{(1,x,y)}{\sqrt{1+x^2+|y|^2}} \in \mathbb{S}^{n,2}\subset\mathbb{R}^{n+1}. \tag{2.16}$$

The image here is exactly the set of points in the sphere with $\omega_0>0$, $\omega_1\ge 1$ and the range space is the closure of the image

$$\mathbb{S}^{n,2} = \left\{\omega\in\mathbb{R}^{n+1}; |\omega|^2 = \omega_0^2+\omega_1^2+|\omega'|^2 = 1, \omega_0\ge 0, \omega_1\ge 0\right\}.$$

The singularity of E_B is at $x=1$, $y=0$ which is in the interior of $\mathbb{S}^{n,2}$. Now consider the decomposition of E_B in terms of the basic homomorphisms which map one dy^α or $dx\wedge dy^\beta$ to another.

LEMMA 5. *The coefficients of $E_B(x,y)$, for $B=A$ or R, in terms of the basic homomorphisms lift to functions on $\mathbb{S}^{n,2}$ which are smooth up to both boundaries and vanish at $\{\omega_0=0\}$ to order $n-1$.*

Notice that $\mathbb{S}^{n,2} \subset \mathbb{R}^{n+1}$ is a compact manifold with corners. Still there is no confusion as to the meaning of smoothness up to the boundary; it just means that the function can be extended across the boundary to be smooth.

PROOF. A typical coefficient in E_A is

$$\begin{aligned} y_j / \left|(x-1,y)\right|^n &= \frac{\omega_j/\omega_0}{\left|(\omega_1/\omega_0 - 1, \omega_j/\omega_0)\right|^n} \\ &= \omega_0^{n-1} \cdot \omega_j / \left|(\omega_1 - \omega_0, \omega_j)\right|^n . \end{aligned}$$

The denominator here does not vanish, and is smooth, near the boundary of $\mathbb{S}^{n,2}$. The other terms are similar. $\square$

Now, I shall *not* proceed to a detailed treatment of the mapping properties of $d+\delta$ acting on the absolute or relative domains on a general compact manifold with boundary. However, recall that it is supposed to proceed much as in the boundaryless case. As briefly discussed above there is an appropriate space of operators, in this case $\Psi_0^{\mathbb{Z},\mathbb{Z}}(X, \Lambda^*)$. It has *two* orders, the first corresponding to the same 'differential' order as before. The second order corresponds to an 'order of vanishing at the boundary'. With each order there is a symbol map which 'delineates the gradation' in the sense that it fixes the operator up to terms one order lower in that sense. The big difference is that the second symbol map takes values in model kernels on $\mathbb{S}^{n,2}$ just as we have been considering. Thus, by combining the interior model from before and the new boundary model we can construct a parametrix and then, by iteration, get to the true generalized inverse. More about this later.

4. Boundary values

Instead let me look at the properties of the Hodge cohomologies, $H^k_{\mathrm{Ho},A}(X)$ and $H^k_{\mathrm{Ho},R}(X)$ related to some metric and the relationship between them.

As in the boundaryless case, but now using Green's formula, integration by parts gives

$$\int_X (|du|^2 + |\delta u|^2) dg = ((d+\delta)u, (d+\delta)u) = 0$$
$$\text{if } u \in H^*_{\mathrm{Ho},B}(X) \text{ for } B = A, R.$$

Thus, harmonic forms satisfying either the relative or absolute boundary conditions are both d- and δ- closed. Let us use the Hodge decomposition on ∂X for the induced metric h to define

$$H^k_{\mathrm{Ho},R}(X) \ni u \xmapsto{[A]} [Au] = [i^*_{\partial X} \iota_\nu u] \in H^{k-1}_{\mathrm{Ho}}(\partial X);$$

that is, just project $i^*_{\partial X} \iota_\nu u$ onto its harmonic part. For absolute Hodge cohomology there is a similar map

$$H^k_{\mathrm{Ho},A}(X) \ni u \xmapsto{[R]} [Ru] = [i^*_{\partial X} u] \in H^k_{\mathrm{Ho}}(\partial X)$$

which projects $i^*_{\partial X} u$ onto its harmonic part. Let us note an important relationship between these two maps.

LEMMA 6. *If* $u \in H^{k+1}_{\mathrm{Ho},R}(X)$ *and* $v \in H^{k}_{\mathrm{Ho},A}(X)$ *then* $\delta A u = 0$, $dRv = 0$ *and*

$$([A]u, [R]v) = 0,$$

i.e. the respective images in boundary cohomology are orthogonal.

PROOF. Suppose $u \in H^*_{\mathrm{Ho},R}(X)$ then from the formula for δ,

$$0 = A\delta u = -\delta A u\,.$$

Similarly if $v \in H^*_{\mathrm{Ho},A}(X)$ then Ru is closed. By definition, $[A]u$ is the harmonic part of Au and $[R]v$ is the harmonic part of Rv so from the Hodge decomposition on ∂X

$$\begin{aligned} ([A]u, [R]v) &= (Au, Rv) \\ &= ((d+\delta)u, v) - (u, (d+\delta)v) \\ &= 0 \end{aligned}$$

by Green's formula. This proves the lemma. $\square$

As well as the maps $[R]$ and $[A]$, consider the orthogonal projections, given by the absolute and relative Hodge decomposition respectively

$$\begin{aligned} I &: H^*_{\mathrm{Ho},R}(X) \longrightarrow H^*_{\mathrm{Ho},A}(X) \\ J &: H^*_{\mathrm{Ho},A}(X) \longrightarrow H^*_{\mathrm{Ho},R}(X). \end{aligned}$$

PROPOSITION 10. *The maps* $[R]$, $[A]$, I *and* J

$$\begin{tikzcd}
 & H^*_{\mathrm{Ho}}(\partial X) & \\
H^*_{\mathrm{Ho},R}(X) \arrow[ur, "{[A]}"] \arrow[rr, shift left, "I"] & & H_{\mathrm{Ho},A}(X) \arrow[ul, "{[R]}"'] \arrow[ll, shift left, "J"]
\end{tikzcd}$$

satisfy the following relations:

$$I^* = J,\ \mathrm{Ran}(I) = \mathrm{Nul}([A]),\ \mathrm{Ran}(J) = \mathrm{Nul}([R]) \tag{2.17}$$

$$\mathrm{Ran}([R]) \oplus \mathrm{Ran}([A]) = H^*_{\mathrm{Ho}}(\partial X). \tag{2.18}$$

PROOF. Green's formula shows that if $u \in H^*_{\mathrm{Ho},R}(X)$ and $v \in \mathrm{Dom}^*_A(X)$ then

$$(u, \delta v) = (du, v) + (Ru, Av) = 0$$

Similarly if $u \in H^*_{\mathrm{Ho},A}(X)$ and $v \in \mathrm{Dom}^*_R(X)$ then $(u, dv) = 0$. Thus, using the Hodge decompositions,

$$\begin{aligned} H^*_{\mathrm{Ho},R}(X) &\subset d\,\mathrm{Dom}^*_A(X) \oplus H^*_{\mathrm{Ho},A}(X) \text{ and} \\ H^*_{\mathrm{Ho},A}(X) &\subset \delta\,\mathrm{Dom}^*_R(X) \oplus H^*_{\mathrm{Ho},R}(X). \end{aligned} \tag{2.19}$$

Now if $u \in H^*_{\mathrm{Ho},R}(X)$ and $v \in H^*_{\mathrm{Ho},A}(X)$ then

$$(Iu, v) = (u, v) = (u, Jv)$$

showing that $J = I^*$.

If $u \in H^*_{\mathrm{Ho},R}(X)$ then

$$u = Iu + dv, v \in \mathrm{Dom}^*_A(X) \Longrightarrow 0 = Ru = RIu + d(Rv)$$

shows that $[R] \circ I = 0$ so $\mathrm{Ran}(I) \subset \mathrm{Nul}([R])$. To see the converse, suppose that $v \in H^*_{\mathrm{Ho},A}(X)$ and $[R]v = 0$. Since $dRv = 0$, this implies that $Rv = d\gamma$ for some smooth form γ on ∂X. Consider a cut-off function $\rho \in \mathcal{C}^\infty([0,1)$ which is identically equal to 1 near 0. If the support of ρ is small enough then $\rho(x) \in \mathcal{C}^\infty(X)$ is 1 near ∂X and $Rd(\rho(x)\gamma) = d\gamma$. Thus $R(v - d(\rho(x)\gamma) = 0$; in view of this the relative decomposition

$$v - d(\rho\gamma) = u + d\kappa,\ \ u \in H^*_{\mathrm{Ho},R}(X),\ \ w \in \mathrm{Dom}^*_R(X)$$

has no coexact part, by Green's formula. Writing this as

$$v = u + d(\rho\gamma + w)$$

and noting that, from Green's formula, $d(\rho\gamma + w)$ is orthogonal to both $H^*_{\mathrm{Ho},A}(X)$ and $\delta\,\mathrm{Dom}^*_A(X)$ it follows that $v = Iu$. This proves the second relation in (2.17).

The proof of the third equation in (2.17) is similar.

Next suppose that $\gamma \in H^*_{\mathrm{Ho}}(\partial X)$ is in the orthocomplement to $\mathrm{Ran}([A])$. It follows that the form $d(\rho(x)\gamma)$ is orthogonal to $H^*_{\mathrm{Ho},R}(\partial X)$

$$(d(\rho\gamma), u) = (\rho\gamma, \delta u) - (\gamma, Au) = (\gamma, [A]u) = 0.$$

Since it is also orthogonal to $\delta\,\mathrm{Dom}^*_R(X)$,

$$(d(\rho\gamma), \delta w) = (\gamma, A\delta w) = -(\gamma, \delta A w) = 0,$$

it is necessarily of the form dv for some $v \in \mathrm{Dom}^*_R(X)$. Thus $\rho(x)\gamma - v$ is a closed form. Taking its absolute decomposition

$$\rho\gamma - u = u' + d\lambda,\ \ u' \in \mathrm{Dom}^*_A(X),\ \ u \in H^*_{\mathrm{Ho},A}(X),$$

since any closed smooth form is orthogonal to $\delta\,\mathrm{Dom}^*_A(X)$. Applying the pull-back operator R shows that

$$\gamma = Ru + dR\lambda \Longrightarrow \gamma = [R]u$$

which is the desired conclusion, showing that $\mathrm{Ran}([A])^\perp = \mathrm{Ran}([R])$. $\square$

These relationships between the maps I, J, $[A]$ and $[R]$ actually just reduce to the statement that

$$\cdots \longrightarrow H^{k-1}_{\mathrm{Ho}}(\partial X) \xrightarrow{[A]^*} H^k_{\mathrm{Ho},R}(X) \xrightarrow{I} H^k_{\mathrm{Ho},A}(X) \xrightarrow{[R]} \cdots \tag{2.20}$$

is a long exact sequence. It is the Mayer-Vietoris sequence for X relative to its boundary.

5. deRham theorems

For later reference, I will examine some of the alternative deRham theories giving either the absolute or relative cohomology of the compact manifold with boundary X. The basic deRham complexes have been introduced above in (2.1) and (2.2). Suppose that a fibration of the manifold around the boundary has been chosen, that is there is a neighbourhood O of ∂X and a diffeomorphism

$$c : O \longleftrightarrow [0, \epsilon) \times \partial X. \tag{2.21}$$

Shrinking the neighourhood, and ϵ, if necessary a global defining function $\rho \in \mathcal{C}^\infty(X)$ can be chosen extending c^*x, where x is the coordinate in the factor of $[0, \epsilon)$. Let c_∂ denote the second component of c, i.e. the projection onto ∂X.

Then consider the spaces of forms

$$A_c(X; \Lambda^*) = \{u \in \mathcal{C}^\infty(X; \Lambda^*); u = c^*_\partial\phi,\ \ \phi \in \mathcal{C}^\infty(\partial X; \Lambda^*) \text{ near } \partial X\} \text{ and} \tag{2.22}$$

$$R_c(X; \Lambda^*) = \{u \in \mathcal{C}^\infty(X; \Lambda^*); u = d\rho \wedge c^*_\partial\psi,\ \ \psi \in \mathcal{C}^\infty(\partial X; \Lambda^*) \text{ near } \partial X\}\,. \tag{2.23}$$

Both are closed under the action of d.

THEOREM 7. *The action of d on the spaces in* (2.23) *has homology respectively the absolute and relative cohomology of X.*

PROOF. I shall use a retraction argument to show that the complexes in (2.23) are quasi-isomorphic to those in (2.1) and (2.2).

Let $\mu \in \mathcal{C}^\infty([0,\epsilon))$ be a smooth cut-off function which is 1 on $[0,\frac{1}{2}\epsilon)$ and identically 0 on $[0,\frac{1}{4}\epsilon)$. Then on the product space $[0,\epsilon)\times\partial X$

$$M(x,y)=(\mu(x)x,y)$$

defines a smooth map which is the identity near $x=\epsilon$ and for $x<\frac{1}{4}\epsilon$ projects (x,y) to $(0,y)$. Using c it therefore defines a smooth map on X which is the identity outside the neighbourhood in (2.21). The map

$$\mathcal{C}^\infty(X;\Lambda^*)\nu u\longmapsto\psi^*u \tag{2.24}$$

has image in $A_c(X;\Lambda^*)$ for a smaller ϵ. Furthermore ψ^*u is closed or exact with u. A similar retraction map shows that the homology of $A_c(X;\Lambda^*)$ is independent of ϵ so all these deRham cohomologies are naturall isomorphic to the absolute deRham cohomology.

The same retraction applied to $R_c(X;\Lambda^*)$ shows it to be quasi-isomorphic to forms with support in the interior of X, so its gives the relative deRham cohomology. □

6. Abstract Hodge Decomposition

I shall simply abstract from this case of a compact manifold with boundary conditions which suffice to give a Hodge decomposition of forms. Let g be the metric on a manifold X° which has no boundary, but will generally not be compact. We can think of g as providing a 'metric' compactification of X° if, as in the cases we consider, there are no long geodesics which leave every compact subset of X°.

Let $\mathrm{Tar}(X)\subset L^2_g(X^\circ;\Lambda^*)=L^2_g(X;\Lambda^*)$ be the 'target space' – this is the space for which we will get a Hodge decomposition. We require

$$\begin{gathered}\mathrm{Tar}(X)\subset L^2_g(X^\circ;\Lambda^*X)\text{ is dense and graded:}\\ \mathrm{Tar}(X)=\bigoplus_k\mathrm{Tar}^k(X)\text{ where }\mathrm{Tar}^k(X)=\mathrm{Tar}(X)\cap L^2_g(X^\circ;\Lambda^k).\end{gathered} \tag{2.25}$$

Similarly let $\mathrm{Dom}(X)$ be the domain we have found for $d+\delta$. We again require that

$$\mathrm{Dom}(X)\subset\mathrm{Tar}(X)\subset L^2_g(X^\circ;\Lambda^*)\text{ is dense and graded.} \tag{2.26}$$

The most important analytic condition we demand is that

$$d+\delta:\mathrm{Dom}(X)\longrightarrow\mathrm{Tar}(X)\text{ be self-adjoint and Fredholm.} \tag{2.27}$$

The Fredholm condition is the requirement that $d+\delta$ have finite-dimensional null space and range with finite-dimensional complement. The self-adjointness condition is with respect to these spaces, i.e. the requirement that

$$\begin{aligned}(u,(d+\delta)v)&=((d+\delta)u,v)\ \forall\ u,v\in\mathrm{Dom}(X)\text{ and}\\ \mathrm{Dom}(X)&=\big\{u\in\mathrm{Tar}(X);\text{ for some }w\in\mathrm{Tar}(X),\\ &\qquad(u,(d+\delta)v)=(w,v)\ \forall\ v\in\mathrm{Dom}(X)\big\}\,.\end{aligned}$$

Finally it is necessary to demand the orthogonality condition

$$d\,\mathrm{Dom}(X) \perp \delta\,\mathrm{Dom}(X). \tag{2.28}$$

Here all pairings are in $L^2_g(X;\Lambda^*X)$. The action of $d+\delta$ in (2.27) is distributional, i.e. using the fact that $L^2_g(X;\Lambda^*X) \subset \mathcal{C}^{-\infty}(X^\circ;\Lambda^*X)$. Notice that the assumption that $\mathrm{Dom}(X)$ is graded, with respect to form degree, implies that[4]

$$d : \mathrm{Dom}(X) \longrightarrow \mathrm{Tar}(X) \subset L^2_g(X;\Lambda^*)$$

and similarly for δ; this shows that (2.30) is meaningful.

Let me first note that these conditions *do* lead to a Hodge decomposition.

PROPOSITION 11. *If* $\mathrm{Dom}(X)$, $\mathrm{Tar}(X)$ *satisfy (2.25) through (2.30) then*

$$\mathrm{Tar}^k(X) = d\,\mathrm{Dom}^{k-1}(X) \oplus \delta\,\mathrm{Dom}^{k+1}(X) \oplus H^k_{\mathrm{Ho}}(X) \tag{2.29}$$

where

$$H^k_{\mathrm{Ho}}(X) = \left\{\varphi \in \mathrm{Dom}^k(X);\ d\varphi = \delta\varphi = 0\right\}. \tag{2.30}$$

PROOF. The Fredholm assumption in (2.27) shows that

$$\mathrm{Tar}(X) = (d+\delta)\,\mathrm{Dom}(X) \oplus U \tag{2.31}$$

where U is the orthocomplement to the range, i.e.

$$U = \{u \in \mathrm{Tar}(X); (u,(d+\delta)v) = 0 \quad 0\forall\ v \ni v \in \mathrm{Dom}(X)\}.$$

The self-adjointness condition then implies that

$$U = \{u \in \mathrm{Dom}(X); (d+\delta)u = 0\}.$$

The orthogonality condition (2.30) then ensures that $du \perp \delta u$ so $du = \delta u = 0$ separately. Similarly writing (2.31) out gives (2.29) in view of (2.30) and the assumed grading of the spaces. □

To see what is involved in the self-adjointness, consider the two spaces

$$\begin{aligned} \mathrm{Max}(X) &= \{u \in \mathrm{Tar}(X); du, \delta u \in \mathrm{Tar}(X)\} \text{ and} \\ \mathrm{Min}(X) &= \big\{u \in \mathrm{Max}(X);\ \forall\ v \in \mathrm{Max}(X) \\ &\qquad (u,v) = (u,\delta v) \text{ and } (\delta u, v) = (u, dv)\big\}. \end{aligned} \tag{2.32}$$

Using the assumed self-adjointness

$$\mathrm{Min}(X) \subset \mathrm{Dom}(X) \subset \mathrm{Max}(X). \tag{2.33}$$

Moreover, on the space $\mathrm{Bou}(X) = \mathrm{Max}(X)/\mathrm{Min}(X)$

$$\begin{aligned} &\mathrm{Bou}(X) \times \mathrm{Bou}(X) \ni ([u],[v]) \longmapsto \\ &b([u],[v]) = ((d+\delta)u, v) - (u,(d+\delta)v) \end{aligned} \tag{2.34}$$

defines[5] an antisymmetric and nondegenerate bilinear form. The self-adjointness condition is then just the requirement that

$$\mathrm{Lam}(X) = \mathrm{Dom}(X)/\mathrm{Min}(X) \subset \mathrm{Bou}(X) \tag{2.35}$$

be Lagrangian with respect to this 'symplectic' form. Of course, we also require it to be graded and to give rise to a Fredholm operator as in (2.27).

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[4]If $u \in \mathrm{Dom}(X)$ then $u = \sum_k u_k$ with each pure form $u_k \in \mathrm{Dom}^k(X)$. Then $(d+\delta)u \subset \mathrm{Tar}^{k-1}(X) \oplus \mathrm{Tar}^{k+1}(X)$ using (2.25) and $du, \delta u \in \mathrm{Tar}(X)$ separately.

[5]Here $u \in \mathrm{Max}(X)$ is a representative of $[u] \in \mathrm{Bou}(X)$.

This still leaves the orthogonality condition (2.28). As we shall see, this is very closely related to the self-adjointness. Indeed, δ is (by definition) the formal adjoint of d. Suppose that we consider the pairing (Green) form

$$G\left([u],[v]\right)=(du,v)-(u,\delta v),\ \ [u],[v]\in\mathrm{Bou}(X) \tag{2.36}$$

then the symplectic form is just its antisymmetric part[6]

$$b\left([u],[v]\right)=G\left([u],[v]\right)-G\left([v],[u]\right).$$

In terms of this form the condition on $\mathrm{Lam}(X)$, determining the self-adjointness property of $\mathrm{Dom}(X)$, becomes,

$$\mathrm{Lam}(X)=\{[u]\in\mathrm{Bou}(X);G([u],[v])=G([v],[u])\ \forall\ v\in\mathrm{Lam}(X)\} \tag{2.37}$$

The orthogonality condition can be written in terms of the form $\tilde{G}u,v)=(du,\delta v)$ which is defined on $\mathrm{Max}(X)\times\mathrm{Max}(X)$ and which we require to vanish on $\mathrm{Dom}(X)\times\mathrm{Dom}(X)$.

In the case of a compact manifold without boundary we can take

$$\mathrm{Dom}(X)=\mathrm{Tar}(X)=\mathcal{C}^\infty(X;\Lambda^*)$$

to get the $\mathcal{C}^\infty$ Hodge decomposition and, using a Sobolev space,

$$\mathrm{Dom}(X)=H^1(X;\Lambda^*),\mathrm{Tar}(X)=L^2(X;\Lambda^*)$$

to get the L^2 decomposition.

For a compact manifold with boundary we have taken

$$\mathrm{Tar}(X)=\mathcal{C}^\infty(X;\Lambda^*)$$

and we observe straightaway that

$$\mathrm{Max}(X)=\mathrm{Tar}(X),\mathrm{Min}(X)=x\mathcal{C}^\infty(X;\Lambda^*). \tag{2.38}$$

The 'boundary data' space is then

$$\mathrm{Bou}(X)=\mathrm{Max}(X)/\mathrm{Min}(X)=\mathcal{C}^\infty(\partial X;\Lambda^*)\oplus\mathcal{C}^\infty(\partial X;\Lambda^*). \tag{2.39}$$

Green's formula computes the following boundary pairing

$$G(u,v)=-(Ru,Av)$$

when R is projection onto the first factor in (2.39) and A onto the second.

So far we have considered the two domain spaces

$$\begin{aligned}\mathrm{Dom}_R(X)&=\{u\in\mathcal{C}^\infty(X;\Lambda^*);Ru=0\}\ \text{ and}\\ \mathrm{Dom}_A(X)&=\{u\in\mathcal{C}^\infty(X;\Lambda^*);Au=0\}.\end{aligned}$$

I have shown (under the assumption that the metric is a true product near the boundary) that both give rise to self adjoint Fredholm operators $d+\delta$. Furthermore the orthogonal condition is satisfied since

$$\begin{aligned}Rdu&=dRu\\ A\delta u&=-\delta Au\end{aligned}$$

so

$$\tilde{G}(u,v)=-(Ru,A\delta v)=(Ru,\delta Av)=0$$

in either case.

[6]Notice that $\mathrm{Max}(X)$ is graded and hence so is $\mathrm{Min}(X)$ and $(du,v)=(u,\delta v)$ if $u\in\mathrm{Min}(X)$ and $v\in\mathrm{Max}(X)$ (or vice versa). Thus G vanishes on $\mathrm{Min}(X)$ so defines a form on $\mathrm{Bou}(X)$.

Let us consider other possibilities which might give rise to Hodge decompositions. We can get the self-adjointness by combining the absolute and relative boundary conditions in different degrees. Suppose we choose a subset $P \subset \{1, \dots, n-1\}$ and define $\mathrm{Dom}_P(X) \subset \mathcal{C}^\infty(X; \Lambda^* X)$ by requiring

$$(Ru)_k = 0, \ k \in P \qquad (2.40)$$
$$(Au)_k = 0, \ k \notin P.$$

Certainly the self-adjointness holds, since the pairing

$$G(u, v) = -\sum_k ((Ru)_k, (Av)_k)$$

vanishes if $u, v \in \mathrm{Dom}_P(X)$. On the other hand computing the second form

$$\tilde{G}(u, v) = \sum_{k \notin P} ((Ru)_k, \delta(Av)_{k+1}) .$$

There is no reason for the first term vanish. so for the form to be trivial we need

$$(Av)_{k+1} = 0 \ \forall \ v \in \mathrm{Dom}_P(X) \text{ when } k \notin P. \qquad (2.41)$$

Comparing (2.40) and (2.41) we need

$$k + 1 \notin P \ \forall \ k \notin P \Longleftrightarrow P = \{k; 1 \le k < p\} \text{ for some } p.$$

Thus, assuming we show the Fredholm property, we get different decompositions depending on the choice of *perversity*[7] p. The extreme cases are $p = 0$, giving the absolute cohomology and $p = n - 1$ giving the relative cohomology. It will be convenient to allow p to be a real number (generally non-negative) but this does not effect the range of possibilities.

Exercise: Show that for each real p the domain $\mathrm{Dom}_p(X)$ does give a Fredholm boundary condition, i.e.

$$d + \delta : \mathrm{Dom}_p(X) \longrightarrow \mathcal{C}^\infty(X; \Lambda^*)$$

is Fredholm.

I think it is fair to say that the absolute boundary condition, $p = 0$, has greater 'naturality' than the others based on our preference for d (as being universal) as opposed to δ (which depends on the metric). Thus the absolute boundary condition comes from considering d, as an operator $d : \mathcal{C}^\infty(X; \Lambda^*) \to \mathcal{C}^\infty(X; \Lambda^*)$. Then the adjoint δ should have domain

$$\{u \in \mathcal{C}^\infty(X; \Lambda^*); (dv, u) = (v, \delta u) \ \forall \ v \in \mathcal{C}^\infty(X; \Lambda^*)\} .$$

From the properties of G this fixes $A\varphi = 0$. Although the absolute boundary condition is in this sense 'preferable' we shall see later that (modified forms of) the other perversities do appear rather naturally in the difference metric settings.

Although I have been (and generally will be) discussing the 'smooth' Hodge decomposition it is perhaps more traditional to treat the L^2 theory. Thus one gets a Hodge decomposition of L^2 for each p by setting

$$\begin{aligned}
\mathrm{Tar}(X) &= L^2(X; \Lambda^*) \\
\mathrm{Max}(X) &= \left\{u \in L^2(X; \Lambda^*); du, \delta u \in L^2(X; \Lambda^*)\right\} \\
\mathrm{Dom}_p(X) &= \{u \in \mathrm{Max}(X); (Ru)_k = 0, k < p, (Au)_k = 0 \ k \ge p\} .
\end{aligned}$$

[7] Don't ask me.

One has to do a little elliptic regularity theory to see that the boundary conditions are well-defined.

Let me try to make these issues of domains a little more concrete. We will typically take $\mathrm{Tar}(X)$ to consist of the conormal, let me say *classical* conomal, sections of $\Lambda^* X$ (restricted to be square-integrable with respect to the metric). These sections are determined by the fact that they have expansions of the boundary (possibly with logarithmic terms),

$$u \sim \sum_i x^{s_i} u_i$$

which determines u modulo $\dot{\mathcal{C}}^\infty(X; \Lambda^*)$. The pairing G involves only the u_i as will our choice of domain. In fact because of a 'transversality condition' it will generally be the case that $d + \delta$ is surjective on formal expansions. Thus the determination of $\mathrm{Dom}(X)$ will generally reduce to the understanding of the (formal) null space of $d + \delta$ and the decision as to which elements of it should be 'allowed' into the domain.

CHAPTER 3

Cones, horns and cusps

1. Conic metrics

Now that we have examined the 'standard' class of metrics on a compact manifold with boundary, I want to bring out the fact that this is by no means the only class of metrics which one can consider even when, as here, we restrict attention to incomplete metrics.[1] Next I will discuss metrics which make the boundary 'look like a point', whereas the smooth metrics make the boundary look like a submanifold of codimension one.

Consider a point, p, in a Riemann manifold. Riemannian normal coordinates can be introduced around p and then the metric looks flat to first order, i.e.

$$g_{ij} = \delta_{ij} + O(|x|^2)\,.$$

Introducing polar coordinates, $x = r\omega$, $\omega \in \mathbb{S}^{n-1}$, the metric becomes

$$g = dr^2 + r^2\left(d\omega^2 + dr\cdot\gamma + O(r)\right)$$

where $d\omega^2$ is the round metric on the sphere. The introduction of polar coordinates 'blows up' the point so that the manifold now looks like $[0,1)_r\times\mathbb{S}^{n-1}$ nearby. This is the trivial case of a conic metric on a manifold with boundary.

Definition 1. If X is a manifold with boundary a *conic metric* on X is a smooth symmetric 2-cotensor, g, which is positive-definite in the interior and which takes the form

$$g = dx^2 + x^2 h \quad \text{near } \partial X \tag{3.1}$$

in terms of some product decomposition, with h a smooth symmetric 2-cotensor which restricts to $x = 0$ to give a metric on ∂X.

I will discuss the Hodge theory for such a conic metric. In fact it is convenient, as in the non-degenerate case, to consider metrics which are 'product conic' near the boundary in the sense that (3.1) is refined to

$$g = dx^2 + x^2 h_0\,, \quad \text{near } \partial X\,, \tag{3.2}$$

with h_0 a metric on ∂X. Really I want mainly to talk about the *model case* which is of this type but on the manifold

$$g = dx^2 + x^2 h \quad \text{on } c(Y) = [0,\infty)\times Y \tag{3.3}$$

so $Y = \partial c(Y)$. Of course this manifold is not compact.

There is a fundamental difference between the boundary case previously considered and the conic case, in that the latter is intrinsically global in terms of the cross-section. Thus we cannot replace Y by Euclidean space, but rather we need to *use* the compactness of Y.

[1]Thanks to Sang Chin and Andras Vasy for helpful comments on this section.

The model conic metric (3.3) encourages us to use a slightly different 'weighted' decomposition of the form bundles on $[0,\infty)\times Y = c(Y)$; namely we insert factors of x:

$$\begin{gathered}\Lambda^* c(Y) \cong \Lambda^* Y \oplus \Lambda^* Y \\ \Lambda^k c(Y) \ni u = x^k\varphi + \, dx \wedge x^{k-1}\psi, \;\; \varphi \in \Lambda^k Y \,, \psi \in \Lambda^{k-1} Y \,.\end{gathered} \tag{3.4}$$

This has the effect of removing the conformal factors of x from the form inner products

$$\langle u, u'\rangle_g = \langle \varphi, \varphi'\rangle_h + \langle \psi, \psi'\rangle_h \,.$$

The Riemann density of g is

$$dg = x^{n-1}\, dx\, dh$$

where dh is the Riemann density of h.

The decomposition (3.4) reduces d to a 2×2 matrix of operators on x-dependent sections of $\Lambda^k Y \oplus \Lambda^{k-1} Y$; namely

$$\begin{pmatrix} \frac{d}{x} & 0 \\ \partial_x + \frac{k}{x} & -\frac{d}{x} \end{pmatrix}.$$

The factors of $1/x$ here arise from the dependence of the power on the form degree in (3.4); the extra term k/x has the same origin. If u is a k-form and u' is a $(k+1)$-form this shows that, assuming supports are compact

$$\begin{gathered}(du\,,u') \\ = \int_0^\infty \int_y \left(\frac{1}{x}\langle d\varphi\,,\varphi'\rangle + \langle \partial_x\varphi + \frac{k}{x}\varphi\,,\psi'\rangle - \langle \frac{d}{x}\varphi\,,\psi'\rangle\right) x^{n-1}\, dx\, dh\,.\end{gathered}$$

Assuming that u' (or u) vanishes to all orders at the boundary there are no boundary terms on integration by parts so

$$\begin{aligned}(du\,,u') = (u\,,\delta u') = \int_0^\infty \int_Y \Bigg(&\langle\varphi\,,\frac{\delta}{x}\varphi'\rangle \\ &+\langle\varphi\,,-\partial_x\psi' - (n-1-k)\frac{\psi'}{x}\rangle - \langle\varphi\,,\frac{\delta}{x}\psi'\rangle\Bigg)\, x^{n-1}\, dx\, dh\,.\end{aligned}$$

Thus, as a 2×2 matrix acting on $\Lambda^{k+1}Y \oplus \Lambda^k Y$ the operator δ becomes

$$\begin{pmatrix} \frac{\delta}{x} & -\partial_x - \frac{(n-1-k)}{x} \\ 0 & -\frac{\delta}{x} \end{pmatrix}.$$

Combining these two formulae shows that the action of $d+\delta$ on $\Lambda^k c(Y)$ is represented under the identification (3.4) with the 2×2 matrix D where

$$xD = \begin{pmatrix} d+\delta & -x\partial_x - (n-K-1) \\ x\partial_x + K & -(d+\delta) \end{pmatrix} \text{ on } \Lambda^k Y \oplus \Lambda^{k-1} Y, \tag{3.5}$$

K being the number operator on forms on the boundary.

We are primarily interested in identifying appropriate boundary conditions for D (and hence of course $d+\delta$). As we shall see this reduces to the analysis of the null space of D, so I shall proceed to describe this.

Observe that D is *almost* a differential operator in x with coefficients depending only on $d+\delta$. I say *almost* because it also depends on the form degree k. So, suppose $V \subset \mathcal{C}^\infty(Y\,;\Lambda^* Y)$ is a subspace which is decomposed under its form gradation:

$$V = \oplus V_k\,, \quad V_k = V \cap \mathcal{C}^\infty(Y;\Lambda^k)$$

and in addition V is invariant under $d+\delta$:

$$(d+\delta)V \subset V\,.$$

Then D acts on $\mathcal{C}^\infty\left([0,\infty)_x; V \oplus V\right)$. In particular if we decompose $\mathcal{C}^\infty(Y;\Lambda)$ into such spaces V then the null space of D on the various $\mathcal{C}^\infty\left([0,\infty)_x; V \oplus V\right)$ will, collectivly, span the null space of D.

The obvious examples of such V, and the most important ones, are the harmonic subspaces. Thus if $\varphi \in H^k_{\mathrm{Ho}}(Y)$ is chosen then D acts on sections of the form

$$f(x)x^k\varphi + g(x)\,dx \wedge x^k\varphi\,,\, f,g \in \mathcal{C}^\infty\left((0,\infty)\right)$$

Thus[2]

$$D\begin{pmatrix} f(x)\varphi \\ g(x)\varphi \end{pmatrix} = \begin{pmatrix} (-\partial_x g - \frac{n-k-1}{x} g)\,\varphi \\ \left(\partial_x f + \frac{kf}{x}\right)\varphi \end{pmatrix},$$

so the null forms are spanned by

$$\begin{pmatrix} x^{-k}\varphi \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ x^{-n+k+1}\varphi \end{pmatrix}.$$

As forms on $c(Y)$, harmonic for $d+\delta$, these represent

$$\varphi \quad \text{and} \quad x^{-n+2k+1}\,dx \wedge \varphi\,. \tag{3.6}$$

Thus each harmonic form on Y gives rise to two harmonic forms on $c(Y)$, one in the same degree and one a degree higher.

In deciding on the domain of $d+\delta$ we shall have to decide on what combination of these null solutions, localized near $x = 0$, to admit to the domain. We certainly require the functions so admitted to be square-integrable, with respect to the conic metric. Thus, consider which of these forms is in $L^2_{\mathrm{loc},g}\left(c(Y)\right)$, i.e. is locally square-integrable near 0. Assuming that $0 \neq \varphi \in H^k_{\mathrm{Ho}}(Y)$,

$$\begin{aligned} \int_0^1 \int_Y \langle\varphi,\varphi\rangle_g\,dg = & \int_0^1 \int_Y x^{-2k}\langle\varphi,\varphi\rangle_h x^{n-1}\,dx\,dh \\ & = \|\varphi\|^2 \int_0^1 x^{n-2k-1}\,dx \end{aligned}$$

is finite if and only if $n-2k>0$, i.e. $k<n/2$. Similarly the other form is locally square-integrable if and only if

$$\int_0^1 \int_Y \langle x^{-n+2k+1}\,dx \wedge \varphi\,, x^{-n+2k+1}\,dx \wedge \varphi\rangle dg = \|\varphi\|^2 \int_0^1 x^{-n+2k-1}\,dx < \infty,$$

i.e. $k+1 > \frac{n}{2}$ (since it *is* a k-form).

[2]Notice that the second term is a $(k+1)$-form, not a k-form.

Since k is necessarily an integer the effects of these constraints depends on the parity of the dimension, n, of $c(Y)$. Thus if $\varphi_k \in H^k_{\mathrm{Ho}}(Y)$ then

$$\dim c(Y) \text{ even } \Rightarrow \begin{cases} \varphi_k \in L^2_{loc,g} & \text{iff} \quad k \leq \frac{n}{2} - 1 \\ x^{-n+2k+1}\, dx \wedge \varphi_k \in L^2_{\mathrm{loc},g} & \text{iff} \quad k+1 \geq \frac{n}{2} \end{cases}$$

$$\dim c(Y) \text{ odd } \Rightarrow \begin{cases} \varphi_k \in L^2_{\mathrm{loc},g} & \text{iff} \quad k \leq \frac{n-1}{2} \\ x^{-n+2k+1}\, dx \wedge \varphi_k \in L^2_{\mathrm{loc},g} & \text{iff} \quad k+1 \geq \frac{n+1}{2}\,. \end{cases}$$

The important distinction here is that only when $\dim c(Y)$ is odd can φ and $x^{-n+2k+1}\, dx \wedge \varphi$ be both locally square-integrable, and then only for $k = \frac{n-1}{2}$, in the middle degree for forms on the boundary.

The significance of this arithmetic can be seen from the pairing relation. Suppose $\rho \in \mathcal{C}^\infty_c([0,\infty))$ and $\rho(x) = 1$ in $x < \epsilon$ for some $\epsilon > 0$. Then for the two forms $u = \varphi$ and $v = x^{-n+2k+1}\, dx \wedge u$ associated to $\varphi \in H^k_{\mathrm{Ho}}(Y)$ the localized forms ρu and ρv are mapped into $\dot{\mathcal{C}}^\infty(c(Y))$ by $d+\delta$. Thus the pairing

$$G(u,v) = ((d+\delta)(\rho(x)u)\,,\rho(x)v) - (\rho(x)u\,,(d+\delta)(\rho(x)v)) \tag{3.7}$$

is well-defined. It is independent of ρ; in fact it can be computed from the non-degenerate form of Green's formula as

$$\begin{aligned} &\lim_{\epsilon\downarrow 0} \int_{x>\epsilon} \int_Y \left(\langle (d+\delta)(\rho u), \rho v\rangle - \langle \rho u, (d+\delta)(\rho v)\rangle \right) dg \\ &\qquad = -\int_Y \langle \varphi, \varphi\rangle_h \, dh\,. \end{aligned} \tag{3.8}$$

Between elements arising from orthogonal forms in $H^k_{\mathrm{Ho}}(Y)$ the corresponding pairing vanishes, as it clearly does between u and itself and v and itself.

So far we have only considered elements of the null space of $d+\delta$ arising from harmonic forms on Y. Recall that the non-zero eigenvalues of $d+\delta$ are associated to pairs of forms $\mu \in \mathcal{C}^\infty(Y\,;\Lambda^{k-1})$ and $\lambda \in \mathcal{C}^\infty(Y\,;\Lambda^k)$ satisfying

$$d\mu = \tau\lambda \quad \text{and} \quad \delta\lambda = \tau\mu\,, \tau > 0\,. \tag{3.9}$$

Then $\mu+\lambda$ is an eigenform with eigenvalue τ and $\mu-\lambda$ is an eigenform with eigenvalue $-\tau$.

As discussed above the two-dimensional space spanned by μ and λ generates a four-dimension space over $\mathcal{C}^\infty((0,\infty))$ which is invariant under $d+\delta$, with basis

$$\mu\,,\ dx\wedge\mu\,,\ \lambda\,,\ dx\wedge\lambda\,.$$

Thus there is a corresponding four-dimensional null space of $d+\delta$.[3] From the grading properties of $d+\delta$ there must be two independent solutions of the form

$$f(x)\, dx\wedge\mu + g(x)\lambda \tag{3.10}$$

and two independent solutions of the form

$$f(x)\mu + g(x)\, dx\wedge\lambda\,. \tag{3.11}$$

From the homogeneity of D these solutions are spanned by homogeneous forms. Thus in the second case we can look for values of t such that

$$D\begin{pmatrix} c_1 x^t \mu \\ c_2 x^t \lambda \end{pmatrix} = 0\,.$$

[3] Since it is a first order ordinary differential operator.

Taking into account (3.5) with μ a $(k-1)$-form and λ a k- form this becomes

$$\begin{aligned}\tau c_1 + (-t-(n-k-1))c_2 &= 0\\ (t+k-1)c_1 - \tau c_2 &= 0\end{aligned}$$

i.e. t must be an eigenvalue of

$$\begin{pmatrix} -(k-1) & \tau \\ \tau & -(n-k-1)\end{pmatrix}$$

which means that $-\tau^2 + (t+k-1)(t+n-k-1) = 0$ so

$$t = -\frac{n-2}{2} \pm \left(\tau^2 + \left(\frac{n}{2}-k\right)^2\right)^{1/2}. \tag{3.12}$$

Since $\tau \neq 0$ the corresponding eigenvector can have neither component zero. The null form is then

$$u = c_1 x^{t+k-1}\mu + c_2 x^{t+k}\, dx \wedge \lambda\,. \tag{3.13}$$

The local square-integability of these forms reduces to the finiteness of the integral

$$\int_0^1 x^{2t+n-1}\, dx$$

which is to say $t + \frac{n}{2} > 0$. From (3.12) this reduces to

$$1 \pm \tau^2 + \left(\frac{n}{2}-k\right)^{1/2} > 0\,. \tag{3.14}$$

Thus the positive square-root always gives rise to a locally square-integrable solution whereas the negative root does so only if

$$\tau^2 + \left(\frac{n}{2}-k\right)^2 < 1\,. \tag{3.15}$$

Next consider the null space of $d+\delta$ on forms as in (3.10). Since these are k-forms we look for solutions of

$$D\begin{pmatrix} d_1 x^s \lambda \\ d_2 x^s \mu\end{pmatrix} = x^{s-1}\begin{pmatrix} (\tau d_1 + (-s-(n-k))d_2)\mu \\ ((s+k)d_1 - \tau d_2)\lambda\end{pmatrix} = 0.$$

Thus the solutions correspond to the singular values of s for the matrix

$$\begin{pmatrix} \tau & -s-n+k \\ s+k & -\tau\end{pmatrix}$$

and the corresponding null vectors. That is,

$$s = -\frac{n}{2} \pm \left(\tau^2 + \left(\frac{n}{2}-k\right)^2\right)^{1/2}. \tag{3.16}$$

The corresponding null eigenforms are locally square-integrable if $2s+n > 0$, i.e. in this case the positive root is always admissible but the negative root is *never* admissible.

Consider the pairing between these solutions, in the sense of (3.7). Since μ and λ are orthogonal on Y the pairing vanishes on each of the two dimensional spaces in (3.13) and those just discussed,

$$v = d_1\, x^{s+k-1}\, dx \wedge \mu + d_2 x^{s+k}\lambda\,. \tag{3.17}$$

On the other hand the pairing between solutions in (3.13) and (3.17)

$$((d+\delta)\rho u\,,\rho v)-(\rho u\,,(d+\delta)\rho v)\\ =\left(-c_1\cdot d_1\|\mu\|^2+c_2\cdot d_2\|\lambda\|^2\right)\lim_{\epsilon\downarrow 0}\epsilon^{s+t+n-1} \tag{3.18}$$

Since the left side is finite, the pairing between solutions in (3.13) and (3.17) with the same sign of square-root must vanish[4] whereas that between opposite signs is non-degenerate. More precisely, amongst locally square-integrable solutions those with negative square-root in (3.7) pair in a non-degenerate fashion with the corresponding (i.e. the same μ and λ) solutions in (3.10) with the positive square root in (3.16). All other pairings between square-integrable solutions vanish, except for the harmonic case previously discussed.

Summarizing the computations so far we have described the tempered null space of $d+\delta$ in the model case.

PROPOSITION 12. *If $\mu\in\mathcal{S}'([0,\infty)X,Y;\Lambda^*)$ is a solution of $(d+\delta)\mu=0$ for the conic metric on $c(Y)=[0,\infty)\times Y$ then μ is a finite linear combination of the null solutions in* (3.10), (3.11) *with powers from* (3.12) *and* (3.16).

To prove this I shall detour into a discussion of the Mellin transform, but this will actually quickly lead to a description of the fundamental solutions for $d+\delta$ — which is what we really want.

So, recall that the one-dimensional Fourier transform

$$\begin{aligned}\hat{\eta}(z) &= \int e^{-izt}\eta(t)\,dt\\ \eta(t) &= \frac{1}{2\pi}\int e^{izt}\hat{u}(z)\,dz\end{aligned}$$

has the following properties, amongst others. It gives an isomorphism (Paley-Wiener theorem)

$$\{\eta\in\mathcal{C}_c^\infty(\mathbb{R})\}\overset{\widehat{}}{\longleftrightarrow}\Big\{\hat{\eta}:\mathbb{C}\to\mathbb{C}\text{ entire s.t. }\exists\ A>0\\ \text{for which }\sup_{z\in\mathbb{C}}\exp(-A\,|\mathrm{Im}z|)\left|z^p\frac{d}{dz}^q\hat{\eta}(z)\right|<\infty\ \forall\ p,q\in\mathbb{N}_0\Big\}.$$

and, by conitnuity, an isomorphism (Plancherel theorem)

$$L^2(\mathbb{R})\ni\eta\overset{\widehat{}}{\longleftrightarrow}\hat{\eta}\in L^2(\mathbb{R})\,.$$

Consider the map

$$e:\mathbb{R}\ni t\mapsto x=e^{-t}\in[0,\infty)$$

which maps $\mathbb{R}$ diffeomorphically onto $(0,\infty)$. Pull-back under this map gives two isomorphisms

$$e^*:\mathcal{C}_c^\infty\left((0,\infty)\right)\leftrightarrow\mathcal{C}_c^\infty(\mathbb{R})$$

and

$$e^*:L_b^2\left([0,\infty)\right)\leftrightarrow L^2(\mathbb{R})$$

[4] Of course you may see this directly by computing the eigenvectors.

where $L^2_b([0,\infty))$ is the space of measurable functions on $[0,\infty)$ which are square-integrable with respect to the 'b-measure' dx/x, i.e

$$\int_0^\infty |v(x)|^2 \frac{dx}{x} < \infty \Leftrightarrow v \in L^2_b([0,\infty))\,.$$

The *Mellin transform* is by definition the Fourier transform composed with this pull-back:

$$v_M(z) = \widehat{e^* v}(z) = \int_0^\infty x^{iz} v_M(z)\, dz\,.$$

The Paley-Wiener and Plancherel theorems carry over immediately to the Mellin transform to yield the basic isomorphisms

$$\mathcal{C}^\infty_c((0,\infty)) \ni v \xleftrightarrow{()_M} \Big\{ v_M : \mathbb{C} \to \mathbb{C} \text{ entire s.t.}$$

$$\text{for some } A \text{ fixed and all } p, q \sup_{z\in\mathbb{C}} \left| \exp(-A|\mathrm{Im} z|) z^p \frac{d^q}{dz} v_M(z) \right| < \infty \Big\}.$$

and

$$L^2_b([0,\infty]) \ni v \xleftrightarrow{()_M} v_M \in L^2(\mathbb{R})\,.$$

Directly from the definition of the Fourier transform, for $\eta \in \mathcal{C}^\infty_c(\mathbb{R})$,

$$\begin{aligned} \widehat{\partial_t \eta(z)} &= i\hat{\eta}(z) \\ \widehat{e^{at}\eta(z)} &= \hat{\eta}(z + ia)\,. \end{aligned}$$

The corresponding identities for the Mellin transform are:

$$\begin{aligned} (x\partial_x v)_M(z) &= -iv_M(z) \\ (x^a v)_M &= v_M(z - ia)\,. \end{aligned}$$

Now, I shall use the Mellin transform to find a fumdamental solution for the operator $d+\delta$ on the cone. Using the decomposition (3.4) and the formula (3.5) for the resulting operator we look for a distribution $E_0(x, y, y')$, with values in $\mathrm{Hom}(\Lambda^* Y \oplus \Lambda^* Y)$, such that

$$DE_0 = \partial(x-1) Id_y \tag{3.19}$$

when Id_y is the identity operator on sections of $\Lambda^* Y \oplus \Lambda^* Y$. Let K be the number operator on forms on Y (not $c(Y)$). Then $K\varphi = k\varphi$ if $\varphi \in \mathcal{C}^\infty(Y; \Lambda^k Y)$ and the operator D has been written as

$$D = \frac{1}{x}\begin{pmatrix} d+\delta & -x\partial_x - (n-K-1) \\ x\partial_x + K & -(d+\delta) \end{pmatrix}$$

acting on sections of $\Lambda^* \oplus \Lambda^* Y$.

Equation (3.19) can then usefully be written $xDE_0 = \delta(x-1)\,\mathrm{Id}_y$. Formally taking the Mellin transform we can look instead for $E_M(z)$ satisfying

$$QE_M = \mathrm{Id}_y, \text{ where } Qv_M = (xDv)_M\,. \tag{3.20}$$

In fact it is more convenient to first use the ellipticity of D in $x > 0$ to choose a paramatrix for E_0, that is a distribution E'_0 with compact support in $x > 0$ (near $x = 1$) such that

$$DE'_0 = \delta(x-1)\,\mathrm{Id}_y - F(x)$$

when F is smooth with values in smoothing operators on Y and support near $x=1$.

The use of the paramatrix to 'remove' the singular term reduces the construction to the search for a solution $G(z)$ of

$$QG(z)=(xF)_M \text{ where } \begin{pmatrix} d+\delta & iz-(n-K-1) \\ -iz+K & -(d+\delta) \end{pmatrix}. \tag{3.21}$$

Here, by the Paley-Wiener theorem, the right side is entire in z with values in the smoothing operators, with rapid decreas of all seminorms as $|\mathrm{Re}z|\to\infty$ whilst $\mathrm{Im}z$ remains bounded. Premiultiplying (3.21) by the matrix $\left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ reduces the problem to

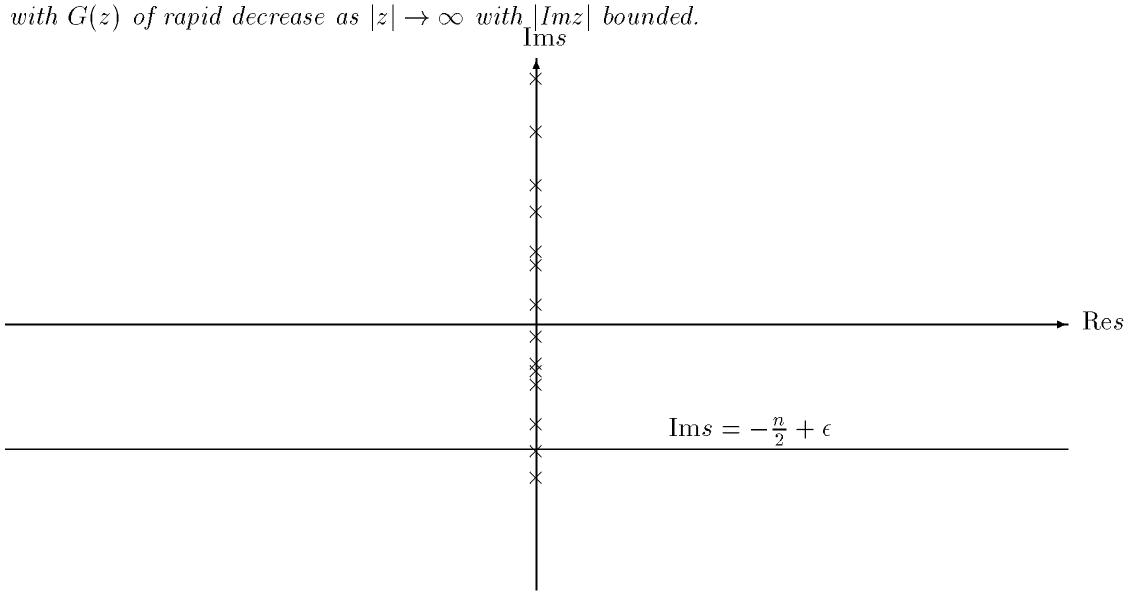
$$\tilde{Q}G(z)=(x\tilde{F})_M,$$
$$\tilde{Q}=\begin{pmatrix} iz-K & d+\delta \\ d+\delta & iz-(n-K-1) \end{pmatrix}, \tilde{F}=\begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}F. \tag{3.22}$$

The singular values of $\tilde{Q}(z)$ are therefore the eigenvalues of the self-adjoint operator on $\Lambda^*Y\oplus\Lambda^*Y$

$$S=\begin{pmatrix} -K & d+\delta \\ d+\delta & -(n-K-1) \end{pmatrix}. \tag{3.23}$$

It is precisely these eigenvalues which we had previously computed. In particular the solution to (3.23) has the following properties

LEMMA 8. *If F is a smooth function of x with support near $x=1$ and values in smoothing operators on $\Lambda^*Y\oplus\Lambda^*Y$ then the solution $G(z)$ to (3.22) is a menomorphic function of z with poles only at the singular values of (3.23), on the pure imaginary axis, only of order 1, with residues finite rank smoothing operators and with $G(z)$ of rapid decrease as $|z|\to\infty$ with $|Imz|$ bounded.*



If $a\in\mathbb{R}$ is such that no singular value has imaginary part a then we can define a distribution from G by taking the inverse Mellin transform. Since the line $\mathrm{Im}s=-\frac{n}{2}$

correspond precisely to square-integrability, we shall choose

$$\begin{aligned} a = -\frac{n}{2} + \epsilon, \epsilon > 0 \text{ so small that} \\ \operatorname{Im} z \notin \left(-\frac{n}{2}, -\frac{n}{2} + 2\epsilon\right) \text{ if } \tilde{Q}(z) \text{ is singular}. \end{aligned} \tag{3.24}$$

Then, as a function of $s \in \mathbb{R}$

$$G(s + ia) \in L^2\left(\mathbb{R}; \mathcal{C}^\infty(Y \times Y; \operatorname{Hom}(\Lambda^* \oplus \Lambda^*)\right)$$

so we can define a kernel G'_a by

$$\left(x^{-a} G'_a\right)_M(s) = G(s + ia). \tag{3.25}$$

For this choice of contour of integration we can easily describe the asymptotic expansion of G'_a.

Proposition 13. *For the choice of a in (3.24), G'_a is a smooth kernel on $(0, \infty) \times Y^2$ with a complete asymptotic expansion at $x = 0$ and $x = \infty$:*

$$G'_a \sim \sum_{\{j; s_j > -\frac{n}{2}\}} x^{s_j} e_j(y, y') \text{ as } x \downarrow 0 \tag{3.26}$$

$$G'_a \sim \sum_{\{j; s_j \le -\frac{n}{2}\}} x^{-s_j} e_j(y, y') \text{ as } x \uparrow \infty, \tag{3.27}$$

where s_j are the constants in

2. Metric Horns

As preparation for the more general models arising in the study of stratified varieties I shall also examine the Hodge theory for a compact manifold with boundary equipped with a metric g which makes a neighbourhood of the boundary into a 'horn'. Thus, in some neighbourhood of the boundary and for some product decomposition

$$g = dx^2 + x^{2a} h \qquad 0 < x < \epsilon, \qquad 0 < a < 1. \tag{3.28}$$

In the 'limiting cases' $a = 0$ and $a = 1$ the non-degenerate and conic cases, as previously considered, appear.

Following the general procedure outlined above we shall investigate possible choices for the domain of $d + \delta$ defined with respect to such a metric and also analyze the 'model' case of a metric globally of the form (3.28) on the product $c(Y)$ for a compact manifold (without boundary for the moment) Y.

The form of the metric in (3.28) can be changed by introducing the singular boundary variable $t = cx^{1-a}$ so that

$$g = t^{2\alpha}(dt^2 + h') \qquad 0 < \alpha < \infty, \ \alpha = \frac{a}{1-a}. \tag{3.29}$$

It is slightly more convenient to work with the 'conformal' metric in (3.29) than the metric in (3.28). Note that this singular transformation changes the $\mathcal{C}^\infty$ structure on the manifold, near the boundary. However it does not change the 'conormal' structure in the sense that $\mathcal{A}_{\text{phg}}(X)$ is unaltered.

The Hodge decompositions we will obtain are for the target space

$$\operatorname{Tar}(X) = \mathcal{A}_{\text{phg}}(X; \Lambda^* X) \cap L^2_g(X; \Lambda^* X), \tag{3.30}$$

so they are not affected by change of variable. The Riemannian measure of the metric (3.28) is[5]

$$dg = t^{\alpha n}\, dt\, dh\,. \tag{3.31}$$

As usual, near the boundary or globally in the model case, it is convenient to 'rescale' the form bundles by identifying sections of $\Lambda^* X$ with sections of $\Lambda^* Y \oplus \Lambda^* Y$ where $Y = \partial X$, using

$$\Lambda^k X \ni u = t^{\alpha k}\varphi + t^{\alpha k}\, dt \wedge \psi\,. \tag{3.32}$$

Then the fibre inner product on $\Lambda^* X$ becomes

$$\langle u, u' \rangle_g = \langle \varphi, \varphi' \rangle_h + \langle \psi, \psi' \rangle_h\,.$$

The condition $u \in \mathcal{A}_{\mathrm{phg}}(X, \Lambda^* X)$ is the requirement that u be $\mathcal{C}^\infty$ in X° and have an expansion near the boundary; this expansion can be written in terms of the tangential forms φ and ψ:

$$\varphi \sim \sum_{i,k} t^{s'_i} (\log t)^k \varphi_{ik}\,,\ \ \psi \sim \sum_{i,k} t^{s''_i} (\log t)^k \psi_{ik}. \tag{3.33}$$

LEMMA 9. *The polyhomogeneous conormal target space* (3.30) *corresponds to those tangential forms with expansions* (3.33) *having*

$$\mathbb{R}e s'_i\,, \quad \mathbb{R}e s''_i > -\frac{1}{2} - \frac{\alpha n}{2}\,. \tag{3.34}$$

PROOF. The condition is the convergence of $\int_0^1 t^{2s+\alpha n}\, dt$. □

Since it is all that really arises below we shall assume that the exponents s'_i, s''_i are real, but this is actually irrelevant.

In terms of the decomposition (3.32), d is represented by the matrix of operators

$$t^{-\alpha} \begin{pmatrix} d & 0 \\ \partial_t + \frac{\alpha K}{t} & -d \end{pmatrix}, \quad K = k \text{ on } \Lambda^k Y\,.$$

The computation of the formal adjoint is given by

$$\begin{aligned}\int_0^1 \int_Y & t^{-\alpha}\langle d\varphi, \varphi' \rangle + t^{-\alpha}\langle \partial_t \varphi + \frac{\alpha k}{t}\varphi, \psi' \rangle - t^{-\alpha}\langle d\psi, \psi' \rangle t^{\alpha n}\, dt\, dh \\ &= \int_0^1 \int_Y t^{-\alpha}\langle \varphi, \delta\varphi' \rangle + t^{-\alpha}\langle \varphi, \left(-\partial_t - \frac{1}{t}\left(\alpha(n-1) - \alpha k\right)\right)\psi' \rangle \\ &\qquad\qquad - t^{-\alpha}\langle \psi, \delta\psi' \rangle t^{\alpha n}\, dt\, dh,\end{aligned}$$

where φ and ψ have supports in $(0,1) \times Y$.

Thus δ is represented by the matrix:

$$t^{-\alpha} \begin{pmatrix} \delta & -\partial_t - \frac{\alpha}{t}(n - K - 1) \\ 0 & -\delta \end{pmatrix}$$

and the operator $d + \delta$ by the matrix

$$D = t^{-1-\alpha} \begin{pmatrix} t(d+\delta) & -t\partial_t - \alpha(n - K - 1) \\ t\partial_t + \alpha K & -t(d+\delta) \end{pmatrix}. \tag{3.35}$$

[5] I will drop the prime for h' in (3.29) and consider it just as some metric h on ∂X.

If we write $\mathcal{A}^{[m]}_{\mathrm{phg}}(X;\Lambda^*X)$ for the subspace of $\mathcal{A}_{\mathrm{phg}}(X;\Lambda^*X)$ consisting of the elements of 'order' at least m in the sense that the expansions in (3.33) only have terms with $\operatorname{Re} s_i' > m$ and $\operatorname{Re} s_i'' > m$ then we find

PROPOSITION 14. *For any $m \in \mathbb{R}$,*

$$d+\delta : \mathcal{A}^{[m]}_{phg}(X;\Lambda^*X)/\dot{C}^\infty(X;\Lambda^*X) \longrightarrow \mathcal{A}^{[m-1-\alpha]}_{phg}(X;\Lambda^*X)/\dot{C}^\infty(X;\Lambda^*X) \tag{3.36}$$

is surjective.

PROOF. Certainly $d+\delta$ acts on $\dot{C}^\infty(X;\Lambda^*X)$, so the map (3.36) is well defined, given the presence of the leading factor $t^{-1-\alpha}$ in (3.35), and the fact that the entries in the matrix clearly preserve order.

Since (3.36) is a map of the formal expansions, to prove its surjectivity it suffices to prove the surjectivity on the quotients

$$d+\delta : \mathcal{A}^{[m]}_{\mathrm{phg}}(X;\Lambda^*X)/\mathcal{A}^{[m+\gamma]}_{\mathrm{phg}}(X;\Lambda^*X) \to \mathcal{A}^{[m-1-\alpha]}_{\mathrm{phg}}(X;\Lambda^*X)/\mathcal{A}^{[m-1-\alpha+\gamma]}_{\mathrm{phg}}(X;\Lambda^*X) \text{ for any one } \gamma > 0\,.$$

Taking $\gamma < 1$ this simplifies the question since the diagonal terms in (3.35) increase order by 1. Thus the problem is reduced to the surjectivity of the map

$$\begin{pmatrix} 0 & -t\partial_t - \alpha(n-K-1) \\ t\partial_t + \alpha K & 0 \end{pmatrix},$$

as a map on formal expansions (3.33), with s_i', $s_i'' > m$. The surjectivity follows from the fact that

$$(t\partial_t + \alpha k) = t^{-\alpha k}(t\partial_t)t^{\alpha k}\,,$$

and the corresponding surjectivity for $t\partial_t$. Thus

$$t\partial_t \cdot t^s(\log t)^k\varphi = st^s(\log t)^k\varphi + kt^s(\log t)^{k-1}\varphi.$$

For $s \neq 0$, $t\partial_t$ is then an isomorphism on forms with a single power. For $s = 0$ it has null space the constants but is still surjective. □

This discussion not only shows the surjectivity, but allows the null space of (3.36) to be readily described. It is useful to refine the statement a little in terms of the Hodge decomposition of ∂X.

Observe that the quotient spaces can be viewed as tensor products:

$$\mathcal{A}^{[m]}_{\mathrm{phg}}(X;\Lambda^*X)/\mathcal{A}^{[m+\gamma]}_{\mathrm{phg}}(X;\Lambda^*X) \cong \left(\mathcal{A}^{[m]}_{\mathrm{phg}}([0,\infty))/\mathcal{A}^{[m+\gamma]}_{\mathrm{phg}}([0,\infty)\right)\otimes\left(\mathcal{C}^\infty(\partial X;\Lambda^*\partial X)\right)^2\,. \tag{3.37}$$

Indeed this is just the decomposition (3.33). We can consider various subspaces in terms of (3.37), much as in the conic case. In terms of the Hodge decomposition of $\mathcal{C}^\infty(X;\Lambda^*\partial X)$ set

$$\begin{aligned} V^{(k)} &= H^k_{\mathrm{Ho}}(\partial X)\oplus 0\,, \quad W^{(k)} = 0 \oplus H^{k-1}_{\mathrm{Ho}}(X) \\ E^{(k)} &= d\mathcal{C}^\infty(\partial X;\Lambda^{k-1}\partial X)\oplus \delta\mathcal{C}^\infty(\partial X;\Lambda^k\partial X) \\ F^{(k)} &= \delta\mathcal{C}^\infty(\partial X;\Lambda^{k+1}\partial X)\oplus d\mathcal{C}^\infty(\partial X;\Lambda^k\partial X)\,. \end{aligned} \tag{3.38}$$

The Hodge decomposition then implies that the sum over k of the sum of the four subspaces

$$\begin{aligned}
\mathcal{V}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes V^{(k)} \\
\mathcal{W}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes W^{(k)} \\
\mathcal{E}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes E^{(k)} \\
\mathcal{F}^{[m,\gamma]}_{(k)}(X) &= \mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) \otimes F^{(k)} \,,
\end{aligned}$$

where $\mathcal{A}^{[m,\gamma]}_{\text{phg}}([0,\infty)) = \mathcal{A}^{[m]}_{\text{phg}}([0,\infty))/\mathcal{A}^{[m+\gamma]}_{\text{phg}}([0,\infty))$, is the whole of (3.37). Furthermore, from the form of D in (3.35) we conclude that:

$$\begin{aligned}
t^{1+\alpha} D &: \mathcal{V}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{W}^{[m,\gamma]}_{(k+1)}(X) \\
t^{1+\alpha} D &: \mathcal{W}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{V}^{[m,\gamma]}_{(k-1)}(X) \\
t^{1+\alpha} D &: \mathcal{E}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{F}^{[m,\gamma]}_{(k-1)}(X) \\
t^{1+\alpha} D &: \mathcal{F}^{[m,\gamma]}_{(k)}(X) \longrightarrow \mathcal{E}^{[m,\gamma]}_{(k+1)}(X) \,.
\end{aligned} \tag{3.39}$$

Since these spaces give a direct sum decomposition of the spaces in (3.37) we conclude from Proposition 14 that each of the maps in (3.39) is surjective. Since this is true for all γ we do in fact get surjectivity on the corresponding spaces of formal series.

COROLLARY 2. *The maps in* (3.39) *are surjective for all* m, γ *(including* $\gamma = \infty$*).*

From this decomposition we can easily find the null space of the map in (3.36).

PROPOSITION 15. *For* $m \ll 0$ *the null space of* $d + \delta$ *as a map on formal expressions in* (3.36) *can be identified with* $\mathcal{C}^\infty(\partial X; \Lambda^* \partial X) \oplus \mathcal{C}^\infty(\partial X; \Lambda^* \partial X)$ *where the map from this 'generalized boundary data' to formal solutions can be written in terms of* (3.38) *as:*

$$\begin{aligned}
V^{(k)} \ni \begin{pmatrix} \varphi \\ 0 \end{pmatrix} &\longmapsto \begin{pmatrix} t^{-\alpha k} \; \varphi \\ 0 \end{pmatrix} \in \mathcal{V}^*_{(k)} \\
W^{(k)} \ni \begin{pmatrix} 0 \\ \psi \end{pmatrix} &\longmapsto \begin{pmatrix} 0 \\ t^{-\alpha(n-k)} \; \psi \end{pmatrix} \in \mathcal{W}^*_{(k)} \\
E^{(k)} \ni \begin{pmatrix} d\varphi \\ 0 \end{pmatrix} &\longmapsto \begin{pmatrix} t^{-\alpha k} \; d\varphi \\ 0 \end{pmatrix} + u'(d\varphi) \in \mathcal{E}^*_{(k)} \\
E^{(k)} \ni \begin{pmatrix} 0 \\ \delta\psi \end{pmatrix} &\longmapsto \begin{pmatrix} 0 \\ t^{-\alpha(n-k)} \, \delta\psi \end{pmatrix} + u''(\delta\psi) \in \mathcal{E}^*_{(k)} \\
F^{(k)} \ni \begin{pmatrix} \delta\varphi \\ 0 \end{pmatrix} &\longmapsto \begin{pmatrix} t^{-\alpha k} \, \delta\varphi \\ 0 \end{pmatrix} + u'(\delta\varphi) \in \mathcal{F}^*_{(k)} \\
F^{(k)} \ni \begin{pmatrix} 0 \\ d\psi \end{pmatrix} &\longmapsto \begin{pmatrix} 0 \\ t^{-\alpha(n-k-2)} \, d\psi \end{pmatrix} + u''\,(d\psi) \in \mathcal{F}^*_{(k)}
\end{aligned} \tag{3.40}$$

where in the last four cases, the extra terms vanish to one order higher than the leading terms.

PROOF. This just reduces to the analysis of the null space of $t^{1+\alpha}D$. This operator is the sum

$$t^{1+\alpha}D = \begin{pmatrix} 0 & -t\partial_t - \alpha(n-K-1) \\ t\partial_t + \alpha K & 0 \end{pmatrix} + t\begin{pmatrix} d+\delta & 0 \\ 0 & -(d+\delta) \end{pmatrix} \tag{3.41}$$

where the first term preserves boundary order and the second raises it by one (at least). The first two forms are annihilated by D. The leading parts of the remaining four forms are annihilated by the first term in (3.41). Thus, by the surjectivity in the corollary above there are solutions of the form indicated.

Conversely, these span all solutions since the leading part of $t^{1+\alpha}D$ in (3.41) must annihilate the leading term of a formal solution and all such terms appear in the list. $\square$

Proposition 15 does not quite give a natural isomorphism, although one can be chosen; the choice does not affect the important Green pairing.

PROPOSITION 16. *If the isomorphism of Proposition 15 from the space consisting of two copies of*

$$\mathcal{C}^\infty(\partial X; \Lambda^* \partial X)$$

to formal null solutions is written $(R \oplus A)^{-1}$ *then for any pair of formal solutions,* u, v

$$\begin{aligned} G(u,v) &= ((d+\delta)(\rho\tilde{u}), \rho\tilde{v}) - (\rho\tilde{u}, (d+\delta)\rho\tilde{v}) \\ &= -(Ru, Av) + (Au, Rv)\,. \end{aligned} \tag{3.42}$$

PROOF. Here, if u and v are formal solutions, $\tilde{u}$ and $\tilde{v}$ are finite sums consisting of all terms up to a sufficiently high order and $\rho \in \mathcal{C}^\infty_c([0,\infty))$ is identically equal to 1 near $t=0$. The pairing in (3.42) is then well-defined. Furthermore, by the pairing relation in the smooth and non-degenerate case,

$$G(u,v) = \lim_{\epsilon \downarrow 0} -(R_\epsilon \tilde{u}, A_\epsilon \tilde{v})_{h_\epsilon} + (A_\epsilon \tilde{u}, R_\epsilon v)_{h_\epsilon}$$

where R_ϵ is pull-back to $t=\epsilon$ and $A_\epsilon = R_\epsilon \cdot \iota_\nu$ where the unit normal vector field is $t^{-\alpha}\delta_t$.

The orthogonality between the various spaces in (3.38) shows that G vanishes except in pairs arising from $V^{(k)} \times W^{(k+1)}$, $E^{(k)} \times F^{(k-1)}$ (in either order since G is always antisymmetric). For these cases working directly from the leading terms given by Proposition 15 we see that $G(u,v)$ is given by (3.42). $\square$

Recall that we are taking the target space to be

$$\operatorname{Tar}(X) = \mathcal{A}_{\mathrm{phg}}^{[-\frac{1}{2}-\frac{\alpha n}{2}]}(X;\Lambda^* X)\,, \tag{3.43}$$

just the intersection of the polyhomogeneous conormal space with $L^2_g(X;\Lambda^* X)$. From the definition of $\operatorname{Max}(X)$, in (2.32), and the properties of D we have

$$\operatorname{Max}(X) \supset t^{1+\alpha}\operatorname{Tar}(X) = \mathcal{A}_{\mathrm{phg}}^{[\frac{1}{2}-\frac{(n-1)\alpha}{2}]}(X;\Lambda^* X)\,. \tag{3.44}$$

We proceed to examine the quotient, which consists of finite expansions in the corresponding range of 'width' $1+\alpha$:

$$Q(X) = \operatorname{Max}(X)/t^{1+\alpha}\operatorname{Tar}(X) \subset \mathcal{A}_{\mathrm{phg}}^{[-\frac{1}{2}-\frac{n\alpha}{2},1+\alpha]}(X,\Lambda^* X)\,. \tag{3.45}$$

Now each $[u] \in Q(X)$ has a representative $u \in \operatorname{Tar}(X)$ with du, $\delta u \in \operatorname{Tar}(X)$. The surjectivity in Proposition"14" means that $[u] = [u']$ where u' has $(d+\delta)u' \equiv 0$, as well as $du', \delta u' \in \operatorname{Tar}(X)$. Thus the terms in Q are determined by the formal solutions of $(d+\delta)u = v$:

$$Q(X) \cong \{[u]; u \in \operatorname{Tar}(X), (d+\delta)u \equiv 0,\ du, \delta u \in \operatorname{Tar}(X)\} \tag{3.46}$$

where the equivalence relation is always modulo $t^{1+\alpha}\operatorname{Tar}(X)$.

The three classes of solutions, $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{E}$, are all form-graded, so $du \equiv 0$ and $\delta u \equiv$ separately; the last condition in (3.46) therefore only arises for solutions in $\mathcal{F}$. The leading term in the k-form component if any formal null form is either $t^{-\alpha k}$ or $t^{-\alpha(n-k)}$ and it is correspondingly tangential or normal. Such terms are

$$\text{square-integrable iff } \begin{cases} k < \frac{n}{2} + \frac{1}{2\alpha} \\ k > \frac{n}{2} - \frac{1}{2\alpha}. \end{cases} \tag{3.47}$$

$$\operatorname{int}^{1+\alpha}\operatorname{Tar}(X) \text{ iff } \begin{cases} k < \frac{n}{2} - 1 - \frac{1}{2\alpha} \\ k > \frac{n}{2} + 1 + \frac{1}{2\alpha}. \end{cases} \tag{3.48}$$

Thus we only need to examine terms with leading tangential k-form parts for $\frac{n}{2} - 1 - \frac{1}{2\alpha} \le h < \frac{n}{2} + \frac{1}{2\alpha}$ and with leading normal parts for $\frac{n}{2} - \frac{1}{2\alpha} < k \le \frac{n}{2} + 1 + \frac{1}{2\alpha}$; since outside these ranges the corresponding terms are either not square-integrable or else are zero in the quotient.

Lemma 10. *"Q" The quotient $Q(X)$ in ((3.46)) consists precisely of the span of the terms in $\mathcal{V}_{(k)}$ and $\mathcal{W}_{(k)}$ for the ranges*

$$\frac{n}{2} - 1 - \frac{1}{2\alpha} \le k < \frac{n}{2} + \frac{1}{2\alpha} \tag{3.49}$$

$$\frac{n}{2} - \frac{1}{2\alpha} < k \le \frac{n}{2} + 1 + \frac{1}{2\alpha}, \tag{3.50}$$

terms in $\mathcal{E}$ with leading tangential and normal parts for the same ranges and terms in $\mathcal{F}$ with leading tangential and normal parts being k-forms for the ranges.

$$\frac{n}{2} - 1 - \frac{1}{2\alpha} \le k < \frac{n}{2} - 1 + \frac{1}{2\alpha} \tag{3.51}$$

$$\frac{n}{2} + 1 - \frac{1}{2\alpha} < k \le \frac{n}{2} + 1 + \frac{1}{2\alpha}. \tag{3.52}$$

Proof. The k-forms in $\mathcal{F}$ are never graded. Since $Q(X)$ must be graded, these are only admissible if the second term in the expansion, which has precisely one additional factor of t, is in $t^{1+\alpha}\operatorname{Tar}(X)$. This gives the altered (and shortened) ranges in (3.51), (3.52). □

Next we compute the boundary pairing on $Q(X)$. That is,

$$B(u,v) = (d\tilde{u}, \tilde{v}) - (\tilde{u}, \delta\tilde{v}) \tag{3.53}$$

where $\tilde{u}, \tilde{v}$ are representatives of $u, v \in Q(X)$. Here we are using the observation that the pairing vanishes on $t^{1+\alpha}\operatorname{Tar}(X)$, so descends to $Q(X)$. We can see this at the same time as computing B since

$$
\begin{aligned}
B(u,v) &= \lim_{\epsilon\downarrow 0}\int_{t>\epsilon} \left(\langle d\tilde{u},\tilde{v}\rangle_g - \langle \tilde{u},\delta\tilde{v}\rangle_g\right)\, dg \\
&= \lim_{\epsilon\downarrow 0}\int_{\partial X} (i_\epsilon^*\tilde{u}, i_\epsilon^* \subset \nu_\epsilon\tilde{u})_{h_\epsilon}\, dh_\epsilon\,,
\end{aligned}
\tag{3.54}
$$

using the non-degenerate boundary pairing. Thus $dh_\epsilon = \epsilon^{(n-1)\alpha}\, dh$, $\nu_\epsilon = \epsilon^{-\alpha}\,\partial_t$ is the unit inward-pointing normal and the inner product for h_ϵ scaled away by (3.32).

In terms of the tangent-normal decomposition we therefore conclude that

$$
B(u,v) = -\sum (\varphi'_{i'k}, \psi''_{i''k})_h \tag{3.55}
$$

where that sum is over those terms satisfying $s_{i'} + s''_{i''} = (n-1)\alpha$. Thus we find that B vanishes except on the span of

$$
\mathcal{V}_{(k)}\times\mathcal{W}_{(k+1)}\,,\mathcal{E}_{(k)}\times\mathcal{F}_{(k-1)}\ \text{ and } \mathcal{E}_{(k-1)}\times\mathcal{F}_{(k)}\,.
$$

Combining these various observations we arrive at the following descriptions of $Q(X)$.

Proposition 17. *"RA" If the identification of the formal null space of $d+\delta$ with $\mathcal{C}^\infty(X;\Lambda^*X)\oplus\mathcal{C}^\infty(\partial X;\lambda^*\partial X)$ given by Proposition 15" is written $R\oplus A$ then B is identified with the boundary pairing*

$$
B(u,v) = -(Ru,Av)_h \tag{3.56}
$$

3. Cusplike metrics

The next class of metrics we consider on a general manifold with boundary has the same form as (3.28) but are 'on the other side' of the conic metrics:

$$
g = dx^2 + x^{2a}h\,,\ 1<a<\infty\,. \tag{3.57}
$$

As in the case of metric horns these can be reduced to a 'variable' conformal multiple of a fixed metric. Namely setting $t = cx^{a-1}$, for an appropriate constant c, gives

$$
g = t^{2\beta}\left(\frac{dt^2}{t^4}+h'\right)\,,1<\beta<\infty\,,\ \beta=\frac{a}{a-1}\,. \tag{3.58}
$$

Notice that this is really the same as in the horn case, except that t has to be replaced by $1/t$ so that the boundary is still at $t=0$ (instead of $t=\infty$).

Now, the identification for forms is analogous to (3.32), except that dt/t^2 has length $t^{-\beta}$, so we decompose the form bundle as

$$
\Lambda^k X \ni u = t^{\beta k}\varphi + t^{\beta k}\frac{dt}{t^2}\wedge\psi\,. \tag{3.59}
$$

Again this reduces the pointwise inner product to

$$
\langle u,u'\rangle_g = \langle\varphi,\varphi'\rangle_h + \langle\psi,\psi'\rangle_h\,. \tag{3.60}
$$

The volume form is $dg = t^{\beta n}\, dt\, dh/t^2$; again I will drop the prime on h.

Lemma 11. *The polyhomogeneous conormal target space*

$$
\operatorname{Tar}(X) = \mathcal{A}_{phg}(X;\Lambda^*X)\cap L^2_g(X;\Lambda^*X) \tag{3.61}
$$

corresponds to those tangential forms with expansions (3.33) having

$$\mathbb{R}es_i', \; \mathbb{R}es_i'' > \frac{1}{2} - \frac{\beta n}{2}. \tag{3.62}$$

PROOF. The condition has changed to $\int_0^1 t^{2s+\beta n} \frac{dt}{t^2} < \infty$. □

From the same sort of computations as before d is represented by

$$t^{-\beta} \begin{pmatrix} d & 0 \\ t^2\partial_t + t\beta K & -d \end{pmatrix}.$$

Similarly δ acts as

$$t^{-\beta} \begin{pmatrix} \delta & t^2\partial_t - \beta t(n-K-1) \\ 0 & -\delta \end{pmatrix}.$$

Thus $d+\delta$ becomes

$$D = t^{-\beta} \begin{pmatrix} d+\delta & -t^2\partial_t - \beta t(n-K-1) \\ t^2\partial_t + t\beta K & -(d+\delta) \end{pmatrix} \tag{3.63}$$

where, as before, K is the number operator on forms on the boundary.

We get an analogue of Proposition 14, but a little care is necessary.

PROPOSITION 18. *The operator D in (3.63) defines an isomorphism for any m*

$$\begin{aligned} D : \mathcal{A}^{[m,\infty)}_{phg} \left([0,\infty); (H^*_{\mathrm{Ho}}(Y))^\perp \oplus (H^*_{Ho}(Y))^\perp \right) \to \\ \mathcal{A}^{[m-\beta,\infty)}_{phg} \left([0,\infty); (H^*_{\mathrm{Ho}}(Y))^\perp \right) \oplus (H^*_{\mathrm{Ho}}(Y))^\perp \end{aligned} \tag{3.64}$$

and surjective map

$$\begin{aligned} D : \mathcal{A}^{[m,\infty)}_{phg} ([0,\infty)) \otimes (H^*_{\mathrm{Ho}}(Y) \oplus H^*_{\mathrm{Ho}}(Y)) \to \\ \mathcal{A}^{[m-\beta+1,\infty)}_{phg} ([0,\infty)) \otimes (H^*_{\mathrm{Ho}}(Y) \oplus H^*_{\mathrm{Ho}}(Y)). \end{aligned} \tag{3.65}$$

PROOF. In (3.64), D is acting on formal forms with values orthogonal to the null space of $H^*_{\mathrm{Ho}}(Y)$, in both factors, so certainly gives a map as indicated. To prove that (3.64) is an isomorphism it suffices to show that the quotient map

$$t^\beta D \text{ is an isomorphism on } \mathcal{A}^{[m,\gamma]}_{\mathrm{phg}} \left([0,\infty); (H^*_{\mathrm{Ho}}(Y) \oplus H^*_{\mathrm{Ho}}(Y))^\perp \right), \tag{3.66}$$

for any $\gamma > 0$. Taking $\gamma < 1$ the off-diagonal terms in (3.63) disappear, so (3.66) follows from the fact that $d+\delta$, the tangential operator, is an isomorphism on $(H^*_{\mathrm{Ho}}(y))^\perp$.

Similarly, in (3.65), the diagonal terms in (3.63) disappear and the off-diagonal terms (which vanish to one order higher) give an isomorphism as discussed in the proof of Proposition 14. □

Notice how different this is from the horn case. There are almost *no* formul solutions.

PROPOSITION 19. *For a cusp-like metric the null space of D, for $m \ll 0$, consists only of the finite dimensional space spanned by*

$$t^{-\beta k} \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \text{ and } t^{-\beta(n-k-1)} \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \text{ for } \varphi \in H^k_{\mathrm{Ho}}(Y). \tag{3.67}$$

PROOF. This is just the null space of D in (3.65), for $m \ll 0$. □

Applying (3.62) these formal solutions are locally square integrable when

$$\begin{cases} k < \frac{n}{2} - \frac{1}{2\beta} & \text{for } t^{-\beta k}\binom{\varphi}{0} \text{ and} \\ k+1 > \frac{n}{2} + \frac{1}{2\beta} & \text{for } t^{-\beta(n-k-1)}\binom{0}{\varphi} \end{cases} \text{ where } \varphi \in H^k_{\mathrm{Ho}}(Y)\,. \tag{3.68}$$

Writing the isomorphism on formal solutions given by (3.67) as $(R \oplus A)^{-1}$ we see, as before, that the Green pairing on formal solutions is

$$\begin{aligned} G(u,v) &= ((d+\delta)\rho\tilde{u}, \rho\tilde{v}) - (\rho\tilde{u}, (d+\delta)\rho\tilde{v}) \\ &= -(Ru, Av)_h + (Au, Rv)_h\,. \end{aligned} \tag{3.69}$$

PROPOSITION 20. *For a cusplike metric with* $\mathrm{Tar}(X)$ *given by (3.61),*

$$(R, A) : \mathrm{Max}(X)/\mathrm{Min}(X) \cong H_{\mathrm{Ho}}^{\frac{n-1}{2}}(X) \oplus H_{\mathrm{Ho}}^{\frac{n-1}{2}}(X)$$

with Green's pairing being the symplectic pairing,

$$G(u,v) = -(Ru, Av) + (Au, Rv)\,. \tag{3.70}$$

PROOF. From (3.68) the only 'overlap' between the two classes of harmonic forms in (3.67) is when

$$-\frac{1}{2a} < k - \frac{n-1}{2} < 1/2a\,. \tag{3.71}$$

Since $1 < a < \infty$ this can only occur when $k = \frac{n-1}{2}$. The computation of the pairing, in (3.70), follows from(3.69). $\square$

The possibilities for the choice of the domain of $d+\delta$, consistent with the desire for a Hodge decomposition, are therefore the same as in the conic case. In particular if n is even there is no choice.

CHAPTER 4

Resolved spaces

I will now discuss the notion of an iterated fibration of the boundary hypersurfaces of a compact manifold with corners. This will turn out to include the 'resolved' version of a stratified pseudomanifold. Before discussing this resolution I shall motivate the notion of an iterated fibration by discussing the total boundary blow up of a manifold with corners.

1. Manifolds with corners

There is no time to go into a complete discussion of manifolds with corners here. Let me remind you of some of the elementary properties by adopting the following (perfectly correct but somewhat awkward) definition.

Definition 2. A manifold with corners is a topological manifold with boundary, X, for which there is a homeomorphism $e : X \longrightarrow \tilde{X}$ into a $\mathcal{C}^\infty$ manifold without boundary, $\tilde{X}$, of the same dimension and such that the image

$$e(X) = \left\{x \in \tilde{X};\ \rho_j(x) \geq 0, j = 1, \dots, K\right\}$$

where the functions $\rho_j \in \mathcal{C}^\infty(\tilde{X})$ have the following independence property:

$$\begin{aligned}&\text{If } \bar{x} \in \tilde{X} \text{ and } I \subset \{1, \dots, K\} \text{ is such that } \rho_j(\bar{x}) = 0\\ &\forall\ j \in I \text{ then } d_j(\bar{x}),\ \ j \in I, \text{ are independent in } T^*_{\bar{x}}\tilde{X}.\end{aligned} \tag{4.1}$$

In fact the space of functions

$$\mathcal{C}^\infty(X) = e^*\mathcal{C}^\infty(\tilde{X}),$$

which is the $\mathcal{C}^\infty$ structure on X, can be determined by the usual local coordinate chart definition, except for the fact that the boundary hypersurfaces

$$H'_j = e^{-1}\left(\{\rho_j = 0\}\right) \tag{4.2}$$

are embedded, as follows automatically from (4.1).

We are mainly interested in compact manifolds with corners, in which case $\tilde{X}$ can be taken to be compact. Let $M_1(X)$ be the set of boundary hypersurfaces, given as the *components* of the H'_j in (4.2). More generally, let $M_k(X)$ be the set of boundary faces of codimension k. Each $F \in M_k(X)$ is necessarily a component of the intersection of k of the elements of $M_1(X)$. Put

$$M_0(X) = \{X\}\,,\ M'(X) = \bigcup_{k\geq 1} M_k(X),\ M(X) = \bigcup_{k\geq 0} M_k(X).$$

A non-degenerate Riemann metric on X is simply the pull-back of a Riemann metric on $\tilde{X}$. There is a very important difference between the case of a compact manifold with boundary and that of the more general case of a compact manifold with corners. Namely, for a general non-degenerate metric, there is no Hodge decomposition of $\mathcal{C}^\infty(X;\Lambda^*X)$. This can be seen most directly in the appearance of harmonic forms which are not smooth up to the corner, despite the fact that they satisfy appropriate boundary conditions.

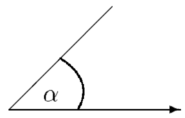
Rather than give any sort of complicated example just consider a sector in $\mathbb{R}^2$ with angle $\alpha < \pi$. Of course this is not compact, but the local behavior of harmonic forms will persist in the compact case. So, simply consider the Laplacian with Dirichlet boundary conditions. In polar coordinates the Laplacian is

$$\Delta = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

It follows that the functions

$$f_k(r,\theta) = r^{\pm\frac{k\pi}{\alpha}} \sin\left(\frac{k\pi\theta}{\alpha}\right) \qquad k \in N \tag{4.3}$$

are harmonic in the sector $0 \le \theta \le \alpha$ and vanish on the boundaries, $\theta = 0, \alpha$.



In general, even for large k, these functions are not smooth down to $r = 0$. An exception is the right angle, $\alpha = \pi/2$, for which the positive signs in (4.3) do give smooth functions.

Thus, in general it is not reasonable to try to find an abstract Hodge decomposition with target $\mathcal{C}^\infty(X;\Lambda^*X)$, even in the case of a non-degenerate metric. In fact we can already suspect from (4.3) that even to get a decomposition with target the space of polyhomogeneous conormal forms it will be necessary to introduce polar coordinates around the corners.

2. Blow-up

I will try not to give a detailed treatment of blow-up! Reasonably complete discussions of real blow up can be found in [**7**] and [**6**]. For present purposes it suffices to consider the blow-up of a boundary face of a compact manifold with corners. This is always possible, giving a new manifold with corners and natural blown-down map

$$[X;F] \xrightarrow{\beta} X \qquad F \in M'(X).$$

If $F \in M_1(X)$ is a boundary hypersurface then $[X;F] = X$ and β is the identity.

In general if $F \in M_k(X)$, $k > 1$, then $[X;F]$ has one more boundary hypersurface $H_F \in M_1([X,F])$ representing the blow-up of F, $\beta(F) = H_F$. The map $\beta : H_F \to F$ is a fibration with fibre diffeomorphic to

$$\mathbb{S}^{k-1,k} = \left\{t \in \mathbb{R}^k; t_j \ge 0 \ \text{ for } 1 \le j \le k, |t| = 1\right\}.$$

The other boundary hypersurfaces of $[X;F]$ can be identified via β with the lifts of the boundary hypersurfaces of X. In all cases the closure of the inverse image

$$\operatorname{clos}\left(\beta^{-1}(H\backslash F)\right) \cong [H; H \cap F]\,,\ H \in M\,(X)\,.$$

3. Total boundary blow-up

If X is a compact manifold with corners we define a new manifold X_{tb} by blowing up *all* the boundary faces of X. There is no point in blowing up the boundary hypersurfaces; it is important to specify the order of the blow-up. We shall always proceed in order of decreasing codimension. Thus if X has maximum boundary codimension k, we first define

$$X_k = [X, M_k(X)]\,,\ \beta_k : X_k \to X.$$

The boundary faces of maximum codimension are manifolds without boundary. There may be several disjoint faces but these lead to independent blow ups, so M_k is well defined. Furthermore all the boundary faces of X of codimension at most $k-1$ lift, in the sense of

$$\operatorname{clos}\beta_k^{-1}(G\backslash \bigcup M_k(X)) = G_k \in M'(X_k),$$

to boundary faces, of the same codimension of X_k. Indeed

$$M_j(X_k) \equiv M_j(X) \quad 1 < j < k.$$

We now proceed iteratively, defining

$$X_l = [X_{l+1},\ M_l(X)]\,,\ k < l < 1,$$

where $M_l(X) \equiv M_l(X_{l+1})$ is the set of boundary faces of codimension l.

The end result of this process of iterated blow-up is the total boundary blow-up, $X_{\mathrm{tb}} = X_2$, which can be written as a repeated blow-up

$$X_{\mathrm{tb}} = [X; M_k(X); M_{k-1}(X); \ldots; M_2(X)]\,.$$

PROPOSITION 21. *The boundary hypersurfaces of X_{tb} are in 1-1 correspondence with the boundary faces of X, with the relationship the total blow-down map β. Two boundary hypersurfaces H, $H' \in M_1(X_{\mathrm{tb}})$ intersect if and only if either $\beta(H) \subset \beta(H')$ or $\beta(H) \cup \beta(H')$.*

PROOF. The correspondence between $M_1(X_{\mathrm{tb}})$ and $M'(X)$, i.e. between the boundary hypersurfaces of X_{tb} and all non-trivial boundary faces of $M(X)$, is essentially by construction. To see the intersection relation, suppose that $F_1, F_2 \in M'(X)$ are such that neither $F_1 \subset F_2$ nor is $F_2 \subset F_1$. Then $F_1 \cap F_2$ is a collection of boundary faces (or is empty). All the faces in the intersection have larger codimension than either F, or F_2. Thus is the construction of X_{tb} the faces in $F_1 \cap F_2$, if any, will be blown-up before either F_1 or F_2. Since F_1 and F_2 are closed, $F_1 \cap F_2$ consists of maximal boundary faces (disjoint if they have different dimensions $G_1, \ldots, G_p$, together with all $F \in M'(X)$ such that $F \subset G_i$ for some i. As the boundary faces of the G_i are blown-up in order of decreasing codimension the structure of the intersection does not change essentially – F_1, F_2 and G_i are replaced by blown up versions but the intersection of the lifts of F_1 and F_2 remains the lift of G_i. When one of the G_i, locally of maximal dimension in the intersection, is blown up the lifts of F_1 and F_2 become disjoint in a neighborhood of the new boundary hypersurface, since they are disjoint in the spherical conormal bundle of G_i. Thus, after the G_i are blown up the lifts of the F_1 and F_2 are disjoint.

Conversely if $F_1 \subset F_2$ then the boundary hypersurface produced by blow-up of F_1 and F_2 are disjoint. $\square$

The space obtained by iterated boundary blow-up of X has more structure than X itself.

PROPOSITION 22. *Each of the boundary hypersurfaces of X_{tb} is a product; if $H_F \in M_1(X_{\mathrm{tb}})$ arises by the blow-up of $F \in M'(X)$ then*

$$H_F \cong F_{\mathrm{tb}} \times \mathbb{S}^{k-1,k}_{\mathrm{tb}}, \ \ F \in M_k(X) \tag{4.4}$$

where $\mathbb{S}^{k-1,k}$ is the intersection of the $(k-1)$-sphere with the totally positive space:

$$\mathbb{S}^{k-1,k} = \left\{ t \in [0,\infty)^k \, ; |t|^2 = 1 \right\}.$$

The product decomposition is realized by the continuous extension of the blow-down map from the interior of H_F.

PROOF. In a neighborhood of the given boundary face F the manifold X is a product, namely

$$X_F \cong F \times [0,\infty)^k \quad \text{near } F \in M_k(X). \tag{4.5}$$

Thus it suffices to work with this local model, the process of blow-up being itself local.[1] The boundary faces of X_F are simply products of boundary faces, so

$$M'(X_F) = \bigcup_k \bigcup_l \quad M_l(F) \times M_{k-l}\left([0,\infty)^k\right).$$

If F has maximum codimension p the faces of X_F of maximal codimension, $p+k$, are the products of these faces with $\{0\}$. Thus the first blow-up gives

$$(X_F)_{p+k} = \left[F \times [0,\infty)^k \, ; M_p(F) \times \{0\} \right].$$

We are interested in a neighborhood of the lift of $F \times \{0\}$ to this new manifold. This is of the form

$$(X_F)_{p+k} \supset X_F^{(p+k)} \cong F_p \times [0,\infty)^k \,,$$

i.e. is just the product of F with its faces of maximal codimension blown up and the model normal space $[0,\infty)^k$.

At the next stage of the iterated blow-up occur the lifts of products $M_p(F) \times M_1\left([0,\infty)^k\right)$ and $M_{p-1}(X) \times \{0\}$. The former are all disjoint from the lift of $F \times \{0\}$, i.e. F, and so only the latter need to be considered. By the same argument as before this shows that there is a neighborhood of the lift of F in

$$(X_F)_{p+k-1} \supset X_F^{(p+k-1)} \simeq F_{p-1} \times [0,\infty)^k \,,$$

just consisting of F with faces of codimension p, and of codimension $p-1$, blown up.

This discussion can be repeated until the codimension of blow-up reaches k. At this point a neighborhood of the lift of F is

$$F_{\mathrm{tb}} \times [0,\infty)^k$$

[1] Local in the manifold which is blown up, not in the blown up manifold!

and the manifolds to be blown up are the lifts of

$$F \times \{0\} \text{ and } M_l(F) \times M_{k-l}\left([0,\infty)^k\right), \text{ for } l \geq 1.$$

The latter lifts are disjoint from the lift of F, so F lifts to a hypersurface and nearby the manifold becomes

$$F_{\text{tb}} \times [0,\infty) \times \mathbb{S}^{k-1,k}$$

with F lifting to $F_{\text{tb}} \times \{0\} \times \mathbb{S}^{k-1,k}$.

For codimensions less than k the only lifts of submanifolds *not* disjoint from the lift of F are the products

$$F_{\text{tb}} \times [0,\infty) \times M'(\mathbb{S}^{k-1,k}).$$

Thus at the end of the sequence of blow-ups the lift of F has a neighborhood

$$F_{\text{tb}} \times [0,\infty) \times \mathbb{S}^{k-1,k}_{\text{tb}}$$

in which the middle variable is the boundary defining function from the lift of F. This proves the proposition. □

Thus we see that the boundary hypersurface $H_F \in M_1(X_{\text{tb}})$ arising from $F \in M_k(X)$ has a product decomposition, which we write as a fibration:

$$\begin{tikzcd} \mathbb{S}^{k-1,k}_{\text{tb}} \arrow[r, dash] & H_F \arrow[d, "\varphi_F"] \\ & F_{\text{tb}}. \end{tikzcd} \tag{4.6}$$

It remains to check how these fibrations are related to each other at points of intersection of the boundary hypersurfaces.

Proposition 23. *If $p \in X_{\text{tb}}$ is a boundary point of codimension q then the boundary hypersurfaces of X_{tb} through p are strictly ordered by the codimension of the boundary faces from which they arise and there is a local product decomposition*

$$X_{\text{tb}} \cong [0,\infty)^q_x \times \mathbb{R}^{n-q}_y \qquad \text{near } p$$

where the boundary hypersurface $\{x_i = 0\}$, $i = 1, \dots, q$ has fibration (4.6) locally having fibres

$$\{(x,y); x_i = 0,\ x_s = \text{const},\ s > i,\ y_l = \text{const},\ l \geq R_i\} \tag{4.7}$$

where R_i is a non-decreasing function of i.

Proof. From Proposition 21 the q hypersurfaces intersecting at p must arise by the blow-up of a chain, under inclusion, of boundary faces of X.. We order the hypersurfaces by increasing codimension, i.e. H_i corresponds to F_i where

$$F_1 \subset F_2 \subset \cdots \subset F_q.$$

Notice that the dimensions are strictly decreasing but may decrease by more than one, i.e. F_l need not be a boundary hypersurface of F_{l-1}, but is necessarily a boundary face of it.

Of course there *is* a complete chain of boundary faces containing $\beta(p) \in X$ which we can write as

$$G_1 \subset G_2 \subset \cdots \subset G_k \ni \beta(p),\ G_l \in M_l(X).$$

Then $F_i = G_{s(i)}$ for some strictly increasing function $s(i)$.

Consider local coordinates $\tilde{x}_1, \ldots, \tilde{x}_k, \tilde{y}_1, \ldots, \tilde{y}_{n-k}$ around $\beta(p)$, where, as usual, the $\tilde{x}_l$'s are local boundary defining functions so that

$$G_i = \{\tilde{x}_l = 0,\ l \leq i\} \qquad \text{near } \beta(p).$$

The lifts of these coordinate functions are smooth near p. Moreover the blow-down map β is a b-map so each $\beta^* \tilde{x}_i$ is the product of a positive function and boundary defining functions near p. For each l, $\tilde{x}_{s(l)}$ vanishes on the F_i for $i \geq l$. Thus if $\rho_j \in \mathcal{C}^\infty(X_{\text{tb}})$ define the hypersurfaces H_j near p,

$$\beta^* \tilde{x}_{s(l)} = a_{s(l)} \prod_{j \geq l} \rho_j,\ 0 < a_{s(l)} \in \mathcal{C}^\infty.$$

This shows that the lifts of the quotients $\tilde{x}_{s(j)}/\tilde{x}_{s(j+1)}$, for $j < q$, and $\tilde{x}_{s(q)}$ itself, can be introduced in place of the ρ_j as defining functions for the boundary hypersurfaces. Having done this, $a_{s(l)} \equiv 1$ (near p) and

$$\rho_j = \beta^* \left(\tilde{x}_{s(j)}/\tilde{x}_{s(j+1)}\right),\ j < q,\ \rho_q = \beta^* \tilde{x}_{s(q)}.$$

The other local boundary defining functions $\tilde{x}_j$, $j \neq s(i)$, have similar decompositions. Thus if we consider the quotients

$$\beta^* \left(\tilde{x}_l/\tilde{x}_{s(j)}\right) \qquad s(j) < l < s(j+1) \tag{4.8}$$

we obtain positive smooth functions. Let us denote by y_r these functions, with a constant subtracted to make them vanish at p and relabeled so that $1 \leq r \leq k - q$. For $r > k - q$ set

$$y_r = \beta^*(\tilde{y}_{r-k}) - \tilde{y}_{r-k}(q) \quad k - q < r \leq n - q.$$

The fact that β is a b-submersion shows that these functions, $x_1, \ldots, x_q, y_1, \ldots, y_{n-q}$ give local coordinates near $p \in X_{\text{tb}}$. By constriction β is then the map

$$\begin{aligned}
\beta(x, y) &= (\tilde{x}, \tilde{y}) \\
\tilde{x}_{s(i)} &= \prod_{l=i}^{i} x_l, \qquad i < q \\
\tilde{x}_{s(q)} &= x_q \\
\tilde{x}_j &= y_{T(j)} \prod_{l=i}^{q} x_l \quad s(i) < j < s(i+1)(\text{or } s(q) < j < k) \\
\tilde{y}_j &= y_{k-q+j}, \quad j = 1, \ldots, n-k.
\end{aligned}$$

Here $j \mapsto T(j)$ is the map defined by the reordering of the y_r's in the discussion following (4.8). The fibres of β over the interior of H_j are then given by the constancy of the x_l, for $l > j$, the y_r arising from (4.8) with $l > s(j)$ and the y_r for $r > k - q$. This defines R_i and gives 4.7. □

4. Iterated fibrations

The properties of the space X_{tb}, with its boundary hypersurfaces each having a fibration, are typical of the resolved spaces we shall need below, except that the fibrations in that case are all products, with fibres being parts of spheres. Dropping these restrictions we arrive at:

DEFINITION 3. An iterated fibration of the boundary hypersurfaces of a compact manifold with corners X is a fibration (in the sense of manifolds with corners) of each boundary hypersurface

$$\begin{tikzcd} Z_H \arrow[r, dash] & H \arrow[d, "\varphi_H"] \\ & B_H . \end{tikzcd} \tag{4.9}$$

such that each boundary point $p \in X$ has a coordinate neighborhood with coordinates $x_1, \ldots, x_q, y_1, \ldots, y_{n-q}$ in terms of which the fibres of $\varphi_{\{x_i=0\}}$ are locally of the form

$$\{(x,y); x_i = 0,\ x_s = \text{const},\ s > i,\ y_l = \text{const},\ l \leq R(i)\} \tag{4.10}$$

where $R(i)$ is a non-decreasing function of $i \in \{1, \ldots, q\}$.

The iterative properties of this notion are important, since this is the approach we need to take in most proofs.

LEMMA 12. *If X is a manifold with corners with an iterated fibration then each boundary hypersurface of X with fibres of (locally) maximal dimension has a naturally induced structure as a manifold with corners with iterated fibration; the dimensions of the fibres are strictly less than those of H.*

PROOF. The conditions in (4.10) show that if H is a boundary hypersurface with fibres of maximal dimension then in the local coordinate form of the fibrations around a point $p \in H$, H itself must always be the defined by $x_q = 0$, since this corresponds to the largest fibre. It follows from (4.10) that the fibrations of the hypersurfaces meeting H restrict to give fibrations of the boundary hypersurfaces of H, since $x_q = 0$ is necessarily a union of fibres. Furthermore the local structure condition (4.10) then restricts to give the same condition for the induced fibrations of the boundary hypersurfaces of H. The fibre dimensions are less than those of H itself. $\square$

In fact we can improve on this proposition a little by noting that the product decomposition near boundary faces, which is the main property of manifolds with corners, extends to include the fibration. That is, if H has maximal fibre dimension then H has an open neighbourhood $O_H \subset X$ which can be decomposed smoothly as a product, $O_H \equiv [0, \epsilon) \times H$ in such a way that the various fibrations remain consistent.

PROPOSITION 24. *If $H \in M_1(X)$ is a boundary hypersurface with fibres of maximal dimension for an iterated fibration of X then ϕ_H can be extended to a fibration of an open neighbourhood O_H of H, $\tilde{\phi}_H : O_H \longrightarrow Y_H \times [0, \epsilon)$, $\epsilon > 0$ such that*

5. Associated metrics

Reverting again to the case of a manifold with corners, X, and its total boundary blow-up, X_{tb}, consider what happens to a smooth, non-degenerate, metric on X under this 'resolution'. To express this we first introduce the notion of an iteratively conic metric associated to an iterated fibration of a manifold with corners.

As usual we can introduce this notion inductively, over the maximal dimension of the fibres. In case the fibrations are all diffeomorphism, so have zero dimensional fibrations, we take an iteratively conic metric to be a non-degenerate metric, one that extends smoothly as a metric 'across the boundaries'. Now, suppose we have defined the notion when the maximal fibre dimension is at most l. If X is a manifold with corners having an iterated fibration with fibres having dimension up to, but not exceeding, $l+1$ we define an iteratively conic metric on X to be a metric on the interior X°, which restricts to an iterately conic metric in a neighbourhood of all boundary hypersurfaces with fibrations of fibre dimension l or less. Since they are disjoint it suffices to consider the behaviour of the metric near one boundary face, H with fibres of dimension $l+1$. Near H we demand that there is a decomposition as in Proposition 24

$$g = d$$

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