CHAPTER 3

Isotropic and scattering calculi[C.IandS.calculi]

In this chapter many of the general constructions with pseudodifferential operators are carried out in the context of two global calculi on $\mathbb{R}^n$. Partly this is done for the obvious reason, that these calculi and results have interesting applications, and partly it is preparatory to the discussion of the geometric algebras of pseudodifferential operators on a compact manifold without boundary and for the scattering algebra on a compact manifold with boundary. Thus, while this chapter is somewhat interstitial, it is designed to clarify the later discussions by separating the generalities of the construction from the particulars of the calculus involved. It should be noted that in this chapter it is generally the *polyhomogeneous* calculus which is under discussion unless it is explicitly stated to the contrary.

3.1. Isotropic operators[S.Sect.isotropic.calculus]

As noted in the discussion in Chapter 2, there are other sensible choices of the class of amplitudes which can be admitted in the definition of a space of pseudodifferential operators than the basic case of $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ discussed there. One of the smallest such choices is the class which is completely symmetric in the variables x and ξ and consists of the symbols on $\mathbb{R}^{2n}$. Thus, $a \in S^m_\infty(\mathbb{R}^{2n}_{x,\xi})$ satisfies the estimates

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|} \quad [1.2.2000.304] \tag{3.1}$$

for all multiindices α and β. If $m \le 0$ this is in the space $S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$; if $m > 0$ it is not, however,

[1.2.2000.303]

LEMMA 3.1. *For any p and n*

$$S^m_\infty(\mathbb{R}^{p+n}) \subset \begin{cases} \bigcap\limits_{0\ge r\ge m} (1+|x|^2)^r S^{m-r}(\mathbb{R}^p_x;\mathbb{R}^n_\xi) & m \le 0 \\ (1+|x|^2)^{m/2} S^m(\mathbb{R}^p_x;\mathbb{R}^n_\xi), & m > 0. \end{cases} \quad [8.2.1998.98] \tag{3.2}$$

PROOF. This follows from (3.1) and the inequalities

$$1+|x|+|\xi| \le (1+|x|)(1+|\xi|),$$
$$1+|x|+|\xi| \ge (1+|x|)^t(1+|\xi|)^{1-t},\ 0 \le t \le 1.$$

□

In view of these estimates the following definition makes sense.

[1.2.2000.313]

DEFINITION 3.1. *For any $m \in \mathbb{R}$ we define*

$$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n) \subset \langle x\rangle^{m+}\Psi^m_\infty(\mathbb{R}^n) \quad [1.2.2000.314] \tag{3.3}$$

as the subspaces determined by

$$A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \Longleftrightarrow \sigma_L(A) \in S^m_{\mathrm{ph}}(\mathbb{R}^{2n}) \quad [1.2.2000.315]$$
$$A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n) \Longleftrightarrow \sigma_L(A) \in S^m_{\infty}(\mathbb{R}^{2n}). \tag{3.4}$$

As in the discussion of the traditional algebra in Chapter 2 we show the $*$-invariance and composition properties of these spaces of operators by proving an appropriate 'reduction' theorem. Note however that there is a small difficulty here. Namely it might be supposed that it is enough to analyse $I(a)$ for $a \in S^m_\infty(\mathbb{R}^{3n})$. This however is not the case. Indeed the definition above is in terms of left-reduced symbols. If $a \in S^m_\infty(\mathbb{R}^{2n})$ is regarded as a function on $\mathbb{R}^{3n}$ which is independent of one of the variables then it is in general *not* an element of $S^m_\infty(\mathbb{R}^{3n})$. For this reason we consider some more 'hybrid' estimates.

Consider a subdivision of $\mathbb{R}^{3n}$ into two closed regions:

$$R_1(\epsilon) = \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \le \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\} \quad [1.2.2000.316]$$
$$R_2(\epsilon) = \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \ge \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\}. \tag{3.5}$$

If $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ consider the estimates

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma} \begin{cases} \langle(x,y,\xi)\rangle^{m-|\alpha|-|\beta|-|\gamma|} & \text{in } R_1(\frac18) \quad [1.2.2000.317] \\ \langle(x,y)\rangle^{m_+}\langle\xi\rangle^{m-|\gamma|} & \text{in } R_2(\frac18). \end{cases} \tag{3.6}$$

The choice $\epsilon = \frac18$ here is rather arbitrary. However if ϵ is decreased, but kept positive the same estimates continue to hold for the new subdivision, since the estimates in R_1 are stronger than those in R_2 (which is increasing at the expense of R_1 as ϵ decreases). Notice too that these estimates do in fact imply that $a \in \langle(x,y)\rangle^{m_+} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$.

[1.2.2000.318]

Proposition 3.1. *If* $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ *satisfies the estimates* (3.6) *then* $A = I(a) \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *and* (2.58) *holds for* $\sigma_L(A)$.

Proof. We separate a into two pieces. Choose $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $0 \le \chi \le 1$, with support in $[-\frac18, \frac18]$ and with $\chi \equiv 1$ on $[-\frac19, \frac19]$. Then consider the cutoff function on $\mathbb{R}^{3n}$

$$\psi(x,y,\xi) = \chi\left(\frac{|x-y|^2}{1+|x|^2+|y|^2+|\xi|^2}\right). \quad [1.2.2000.319] \tag{3.7}$$

Clearly, ψ has support in $R_1(\frac18)$ and $\psi \in S^0_\infty(\mathbb{R}^{3n})$. It follows then that $a' = \psi a \in S^m_{\mathrm{iso}}(\mathbb{R}^{3n})$. On the other hand, $a'' = (1-\psi)a$ has support in $R_2(\frac19)$. In this region $|x-y|$, $\langle(x,y)\rangle$ and $\langle(x,y,\xi)\rangle$ are bounded by constant multiples of each other. Thus a'' satisfies the estimates

$$\begin{aligned} |D^\alpha_x D^\beta_y D^\gamma_\xi a''(x,y,\xi)| &\le C_{\alpha,\beta,\gamma}|x-y|^{m_+}\langle\xi\rangle^{m-|\gamma|} \\ &\le C'_{\alpha,\beta,\gamma}\langle(x,y,\xi)\rangle^{m_+}\langle\xi\rangle^{m-|\gamma|},\ \operatorname{supp}(a'') \subset R_2(\frac19). \quad [1.2.2000.320] \end{aligned} \tag{3.8}$$

First we check that $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$. On $R_2(\frac19)$ it is certainly the case that $|x-y| \ge \frac19\langle(x,y)\rangle$ and by integration by parts

$$|x-y|^{2p} D^\alpha_x D^\beta_y I(a'') = I(|D_\xi|^{2p} D^\alpha_x D^\beta_y a'').$$

For all sufficiently large p it follows from (3.8) that this is the product of $\langle (x,y)\rangle^{m+}$ and a bounded continuous function. Thus, $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$ is the kernel of an operator in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

So it remains only to show that $A' = I(a') \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$. Certainly this is an element of $\langle x\rangle^{m+}\Psi^m_\infty(\mathbb{R}^n)$. The left-reduced symbol of A' has an asymptotic expansion, as $\xi \to \infty$, given by the usual formula, namely (2.58). Each of the terms in this expansion

$$a_L(A') \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_y D^\alpha_\xi a(x,x,\xi)$$

is in the space $S^{m-2|\alpha|}_\infty(\mathbb{R}^{2n})$. Thus we can actually choose an asymptotic sum in the stronger sense that

$$b' \in S^m_\infty(\mathbb{R}^{2n}),\ b_N = b' - \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-2N}_\infty(\mathbb{R}^{2n})\ \forall\ N.$$

Consider the remainder term in (2.47), given by (2.44) and (2.45). Integrating by parts in ξ to remove the factors of $(x-y)^\alpha$ the remainder, R_N, can be written as a pseudodifferential operator with amplitude

$$r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{i^{|\alpha|}}{\alpha!} \int_0^1 dt(1-t)^N (D^\alpha_\xi D^\alpha_y a)((1-t)x+ty,\xi).$$

This satisfies the estimates (3.6) with m replaced by $m-2N$. Indeed from the symbol estimates on a' the integrand satisfies the bounds

$$|D^\beta_x D^\gamma_y D^\delta_\xi D^\alpha_\xi D^\alpha_y a'((1-t)x+ty,\xi)| \\ \le C(1+|(x+t(x-y)|+|\xi|)^{m-2N-|\beta|-|\gamma|-|\delta|}.$$

In $R_1(\frac{1}{8})$, $|x-y| \le \frac{1}{8}\langle (x,y,\xi)\rangle$ so $|x+t(x-y)|+|\xi| \ge \frac{1}{2}\langle (x,y,\xi)\rangle$ and these estimates imply the full symbol estimates there. On R_2 we immediately get the weaker estimates in (3.7).

Thus, for large N, the remainder term gives an operator in $\langle x\rangle^{\frac{m}{2}-N}\Psi^{\frac{m}{2}-N}_{\text{iso}}(\mathbb{R}^n)$. The difference between A' and the operator $B' \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$, which is R_N plus an operator in $\Psi^{m-2N}_{\infty-\text{iso}}(\mathbb{R}^n)$ for any N is therefore in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus $A \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$. □

This is a perfectly adequate replacement in this context for our previous reduction theorem, so now we can show the basic result.

[8.2.1998.99]

THEOREM 3.1. *The spaces $\Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$ (resp. $\Psi^m_{\text{iso}}(\mathbb{R}^n)$) of isotropic (resp. polyhomogeneous isotropic) pseudodifferential operators on $\mathbb{R}^n$, defined by* (3.4) *form an order-filtered $*$-algebra with residual space $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$ (resp. the same) as spaces of kernels.*

PROOF. The condition that a continuous linear operator A on $\mathcal{S}(\mathbb{R}^n)$ be an element of $\Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$ is that it be an element of $(1+|x|^2)^{m/2}\Psi^m_\infty(\mathbb{R}^n)$ if $m \ge 0$ or $\Psi^m_\infty(\mathbb{R}^n)$ if $m<0$ with left-reduced symbol an element of $S^m_\infty(\mathbb{R}^{2n}_{x,\xi})$:

$$q_l : S^m_\infty(\mathbb{R}^{2n}) \longleftrightarrow \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n). \qquad (3.9)$$

[8.2.1998.101]

Thus A^* has right-reduced symbol in $S^m_\infty(\mathbb{R}^{2n})$. This satisfies the estimates (3.6) as a function of x, y and ξ. Thus Propostion 3.1 shows that $A^* \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, since its left-reduced symbol is in $S^m_\infty(\mathbb{R}^{2n})$. This shows the $*$-invariance. Moreover it also follows that any $B \in \Psi^{m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ has right-reduced symbol in $S^{m'}_\infty(\mathbb{R}^{2n})$. Thus if $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{m'}_{\mathrm{iso}}(\mathbb{R}^n)$ then using this result to right-reduce B we see that the composite opertor has kernel $I(a_L(x,\xi)b_R(y,\xi))$ where $a_L \in S^m_\infty(\mathbb{R}^{2n})$ and $b_R \in S^{m'}_\infty(\mathbb{R}^{2n})$. Now it again follows that this product satisifes the estimates (3.6) of order $m + m'$. Hence, again applying Proposition 3.1, we conclude that $A \circ B \in \Psi^{m+m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. This proves the theorem for $\Psi^*_{\infty-\mathrm{iso}}(\mathbb{R}^n)$.

The proof for the polyhomogeneous space $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ follows immediately, since the symbol expansions all preserve polyhomogeneity. □

One further property of the isotropic calculus that distinguishes it strongly from the traditional calculus is that it is invariant under Fourier transformation.
[21.2.1998.107]

Proposition 3.2. *If $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ (resp. $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$) then $\hat{A} \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ (resp. $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$) where $\widehat{\hat{A}u} = A\hat{u}$ with $\hat{u}$ being the Fourier transform of $u \in \mathcal{S}(\mathbb{R}^n)$.*

The proof of this is outlined in Problem 2.20.

3.2. Scattering operators[S.Scattering.operators]

There is another calculus of pseudodifferential operators which is 'smaller' than the traditional calculus. It arises by taking amplitudes in (2.2) which treat the base and fibre variables symmetrically, but not 'simultaneously.' Thus consider the spaces

$$S^{l,m}_\infty(\mathbb{R}^p_z, \mathbb{R}^n_\xi) = \big\{a \in \mathcal{C}^\infty(\mathbb{R}^{p+n});\\ \sup_{\mathbb{R}^{p+n}} (1+|z|)^{-l+|\alpha|}(1+|\xi|)^{-m+|\beta|}|D^\alpha_z D^\beta_\xi a(z,\xi)| < \infty,\ \forall\ \alpha,\beta\big\}. \tag{3.10}$$
[21.2.1998.113]

Observe that

$$S^{l,m}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi) \subset (1+|z|^2)^{l/2} S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi). \tag{3.11}$$
[21.2.1998.115]

We can then define

$$A \in \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \Longleftrightarrow A = (1+|x|^2)^{l/2}B,\\ B \in \Psi^m_\infty(\mathbb{R}^n) \text{ and } \sigma_L(B) \in S^{0,m}_\infty(\mathbb{R}^n_x,\mathbb{R}^n_\xi). \tag{3.12}$$
[21.2.1998.114]

It follows directly from this definition and the properties of the 'traditional' operators that the left symbol map is an isomorphism

$$\sigma_L : \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \longrightarrow S^{l,m}_\infty(\mathbb{R}^n_x,\mathbb{R}^n_\xi). \tag{3.13}$$
[1.2.2000.300]

To prove that this is an algebra, we need first the analogue of the asymptotic completeness, Proposition 2.3, for symbols in $S^{*,*}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.
[1.2.2000.292]

Lemma 3.2. *If $a_j \in S^{l-j,m-j}_\infty(\mathbb{R}^p,\mathbb{R}^n)$ for $j \in \mathbb{N}_0$ then there exists*

$$a \in S^{l,m}(\mathbb{R}^p,\mathbb{R}^n) \text{ s.t. } a - \sum_{j=0}^N a_j \in S^{l-N,m-N}_\infty(\mathbb{R}^p,\mathbb{R}^n)\ \forall\ N \in \mathbb{N}_0. \tag{3.14}$$
[1.2.2000.293]

Even though there is some potential for confusion we write $a \sim \sum_j a_j$ for a symbol a satisfying (3.14).

PROOF. We use the same strategy as in the proof of Proposition 2.3 with the major difference that there are essentially two different symbolic variables. Thus with the same notation as in (2.54) we set

$$a = \sum_j \phi(\epsilon_j z)\phi(\epsilon_j \xi) a_j(z,\xi)^{[1.2.2000.294]} \tag{3.15}$$

and we proceed to check that if the $\epsilon_j \downarrow 0$ fast enough as $j \to \infty$ then the series converges in $S^{l,m}_\infty(\mathbb{R}^p, \mathbb{R}^n)$ and the limit satisfies (3.14).

The first of the seminorms, for convergence, is

$$A_j = \sup_z \sup_\xi (1+|z|)^{-l}(1+|\xi|)^{-m}\phi(\epsilon_j z)\phi(\epsilon_j\xi)|a_j(z,\xi)|.$$

On the support of this function either $|z| \ge 1/\epsilon_j$ or $|\xi| \ge 1/\epsilon$. Thus

$$\begin{aligned} A_j \le \sup_z \sup_\xi (1+|z|)^{-l+j}(1+|\xi|)^{-m+j}|a_j(z,\xi)| \\ \times \sup_z \sup_\xi (1+|z|)^{-j}(1+|\xi|)^{-j}\phi(\epsilon_j z)\phi(\epsilon_j \xi) \\ \le \epsilon_j^j \sup_z \sup_\xi (1+|z|)^{-l+j}(1+|\xi|)^{-m+j}|a_j(z,\xi)| \end{aligned}$$

The last term on the right is a seminorm on $S^{l-j,m-j}_\infty(\mathbb{R}^p, \mathbb{R}^n)$ so convergence follows by choosing the ϵ_j *eventually* smaller than a certain sequence of positive numbers. The same argument follows, as in the discussion leading to (2.56), for convergence of the series for the derivatives and also for the stronger convergence leading to (3.14). Since overall this is a countable collection of conditions, all can be arranged by diagonalization and the result follows. $\square$

With this result on asymptotic completeness the proof of Theorem 3.1 can be followed closely to yield the analogous result on products. In fact we can also define polyhomogeneous operators. This requires a little work if we try to do it directly. However see (1.97) and Problem 1.17 which encourages us to identify

$$\begin{gathered} \mathrm{RC}^*_p \times \mathrm{RC}^*_n : S^{0,0}_{\mathrm{ph}}(\mathbb{R}^p, \mathbb{R}^n) \longleftrightarrow \mathcal{C}^\infty(\mathbb{S}^{p,1} \times \mathbb{S}^{n,1}), \quad {}^{[1.2.2000.297]} \\ S^{l,m}_{\mathrm{ph}}(\mathbb{R}^p, \mathbb{R}^n) = (1+|z|^2)^{l/2}(1+|\xi|^2)^{m/2} S^{0,0}_{\mathrm{ph}}(\mathbb{R}^p, \mathbb{R}^n),\ l,m \in \mathbb{R}. \end{gathered} \tag{3.16}$$

These definitions are discussed as problems starting at Problem 1.18. Thus we simply define

$$\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n) = \left\{ A \in \Psi^{l,m}_{\infty-\mathrm{sc}}; \sigma_L(A) \in S^{l,m}_{\mathrm{ph}}(\mathbb{R}^n, \mathbb{R}^n) \right\}.^{[1.2.2000.299]} \tag{3.17}$$

[1.2.2000.295]

THEOREM 3.2. *The spaces $\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$ (resp. $\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)$) of scattering (resp. polyhomogeneous scattering) pseudodifferential operators on $\mathbb{R}^n$, form an order-bifiltered $*$-algebra*

$$\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \circ \Psi^{l',m'}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \subset \Psi^{l+l',m+m'}_{\infty-\mathrm{sc}}(\mathbb{R}^n)^{[1.2.2000.296]} \tag{3.18}$$

with residual spaces

$$\bigcap_{l,m} \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) = \bigcap_{l,m} \Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n}).^{[1.2.2000.355]} \tag{3.19}$$

3.3. The residual algebra, $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ [S.Residual.isotropic]

The residual isotropic (and scattering) algebra has two important properties not shared by the residual algebra $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$, of which it is a subalgebra (and in fact in which it is an ideal). The first is that as operators on $L^2(\mathbb{R}^n)$ the residual isotropic operators are compact.
[1.2.2000.309]

Proposition 3.3. *Elements of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ are characterized amongst continous operators on $\mathcal{S}(\mathbb{R}^n)$ by the fact that they extend by continuity to define continuous linear maps*

$$A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{3.20}$$ [1.2.2000.356]

In particular the image of a bounded subset of $L^2(\mathbb{R}^n)$ under an element of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is contained in a compact subset.

Proof. The kernels of elements of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ are in $\mathcal{S}(\mathbb{R}^{2n})$ so the mapping property (3.20) follows.

The norm $\sup_{|\alpha|\le 1} |\langle x\rangle^{n+1} D^\alpha u(x)|$ is continuous on $\mathcal{S}(\mathbb{R}^n)$. Thus if $S \subset L^2(\mathbb{R}^n)$ is bounded and $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ the continuity of $A : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ implies that $A(S)$ is bounded with respect to this norm. The theorem of Arzela-Ascoli shows that any sequence in $A(S)$ has a strongly convergent subsequence in $\langle x\rangle^n \mathcal{C}^0_\infty(\mathbb{R}^n)$ and such a sequence converges in $L^2(\mathbb{R}^n)$. Thus $A(S)$ has compact closure in $L^2(\mathbb{R}^n)$. $\square$

The second important property of the residual algebra is that it is 'bi-ideal' or a 'corner' in the bounded operators on $L^2(\mathbb{R}^n)$. Note that it is not an ideal.
[1.2.2000.310]

Lemma 3.3. *If $A_1,\ A_2 \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and B is a bounded operator on $L^2(\mathbb{R}^n)$ then $A_1BA_2 \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.*

Proof. The kernel of the composite $C = A_1BA_2$ can be written as a distributional pairing

$$C(x,y) = \int_{\mathbb{R}^{2n}} B(x',y')A_1(x,x')A_2(y',y)dx'dy' = (B, A_1(x,\cdot)A_2(\cdot,y)) \in \mathcal{S}(\mathbb{R}^{2n}). \tag{3.21}$$ [1.2.2000.321]

Thus the result follows from the continuity of the exterior product, $\mathcal{S}(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{4n})$. $\square$

In fact the same conclusion, with essentially the same proof, holds for any continuous linear operator B from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

3.4. The ring $\operatorname{Id} + \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ [S.Isotropic.ring]

Recall that a bounded operator is said to have finite rank if its range is finite dimensional. If we consider a bounded operator B on $L^2(\mathbb{R}^n)$ which is of finite rank then we may choose an orthonormal basis f_j, $j = 1, \dots, N$ of the range $BL^2(\mathbb{R}^n)$. The functionals $u \longmapsto \langle Bu, f_j\rangle$ are continuous and so define non-vanishing elements $g_j \in L^2(\mathbb{R}^n)$. It follows that the Schwartz kernel of B is

$$B = \sum_{j=1}^N f_j(x)\overline{g_j(y)}. \tag{3.22}$$ [1.2.2000.311]

If $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then the range must lie in $\mathcal{S}(\mathbb{R}^n)$ and similarly for the range of the adjoint, so the functions f_j and g_j are also in $\mathcal{S}(\mathbb{R}^n)$. Clearly the finite rank elements of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ form an ideal in $\Psi^{\infty}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$.

[1.2.2000.312]

Proposition 3.4. *If* $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *then* $\mathrm{Id}+A$ *has, as an operator on* $L^2(\mathbb{R}^n)$*, finite dimensional null space and closed range which is the orthocomplement of the null space of* $\mathrm{Id}+A^*$*. There is an element* $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *such that*

$$(\mathrm{Id}+A)(\mathrm{Id}+B) = \mathrm{Id}-\Pi_1,\ (\mathrm{Id}+B)(\mathrm{Id}+A) = \mathrm{Id}-\Pi_0^{[1.2.2000.322]} \tag{3.23}$$

where $\Pi_0,\ \Pi_1 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *are the orthogonal projections onto the null spaces of* $\mathrm{Id}+A$ *and* $\mathrm{Id}+A^*$ *and furthermore, there is an element* $A' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *of rank equal to the dimension of the null space such that* $\mathrm{Id}+A+sA'$ *is an invertible operator on* $L^2(\mathbb{R}^n)$ *for all* $s \neq 0$*.*

Proof. Most of these properties are a direct consequence of the fact that A is compact as an operator on $L^2(\mathbb{R}^n)$; nevertheless we give brief proofs.

We have shown, in Proposition 3.3 that each $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is compact. It follows that

$$N_0 = \mathrm{Nul}(\mathrm{Id}+A) \subset L^2(\mathbb{R}^n)^{[1.2.2000.327]} \tag{3.24}$$

has compact unit ball. Indeed the unit ball, $B = \{u\, \mathrm{Nul}(\mathrm{Id}+A)\}$ satisfies $B = A(B)$, since $u = -Au$ on B. Thus B is closed and precompact. Any Hilbert space with a compact unit ball is finite dimensional, so $\mathrm{Nul}(\mathrm{Id}+A)$ is finite dimensional.

Now, let $R_1 = \mathrm{Ran}(\mathrm{Id}+A)$ be the range of $\mathrm{Id}+A$; we wish to show that this is a closed subspace of $L^2(\mathbb{R}^n)$. Let $f_k \to f$ be a sequence in R_1, converging in $L^2(\mathbb{R}^n)$. For each k there exists a unique $u_k \in L^2(\mathbb{R}^n)$ with $u_k \perp N_0$ and $(\mathrm{Id}+A)u_k = f_k$. We wish to show that $u_k \to u$. First we show that $\|u_k\|$ is bounded. If not, then along a subsequent $v_j = u_{k(j)}$, $\|v_j\| \to \infty$. Set $w_j = v_j/\|v_j\|$. Using the compactness of A, $w_j = -Aw_j + f_{k(j)}/\|v_j\|$ must have a convergent subsequence, $w_j \to w$. Then $(\mathrm{Id}+A)w = 0$ but $w \perp N_0$ and $\|w\| = 1$ which are contradictory. Thus the sequence u_k is bounded in $L^2(\mathbb{R}^n)$. Then again $u_k = -Au_k + f_k$ has a convergent subsequence with limit u which is a solution of $(\mathrm{Id}+A)u = f$; hence R_1 is closed. The orthocomplement of the range of a bounded operator is always the null space of its adjoint, so R_1 has a finite-dimensional complement $N_1 = \mathrm{Nul}(\mathrm{Id}+A^*)$. The same argument applies to $\mathrm{Id}+A^*$ so gives the orthogonal decompositions

$$\begin{aligned} L^2(\mathbb{R}^n) &= N_0 \oplus R_0,\ N_0 = \mathrm{Nul}(\mathrm{Id}+A),\ R_0 = \mathrm{Ran}(\mathrm{Id}+A^*)_{[1.2.2000.328]} \\ L^2(\mathbb{R}^n) &= N_1 \oplus R_1,\ N_1 = \mathrm{Nul}(\mathrm{Id}+A^*),\ R_1 = \mathrm{Ran}(\mathrm{Id}+A). \end{aligned} \tag{3.25}$$

Thus we have shown that $\mathrm{Id}+A$ induces a continuous bijection $\tilde{A} : R_0 \longrightarrow R_1$. From the closed graph theorem the inverse is a bounded operator $\tilde{B} : R_1 \longrightarrow R_0$. In this case continuity also follows from the argument above.[1] Thus $\tilde{B}$ is the generalized inverse of $\mathrm{Id}+A$ in the sense that $B = \tilde{B} - \mathrm{Id}$ satisfies (3.23). It only remains to show that $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. This follows from (3.23), the identities in

[1] We need to show that $\|\tilde{B}f\|$ is bounded when $f \in R_1$ and $\|f\| = 1$. This is just the boundedness of $u \in R_0$ when $f = (\mathrm{Id}+A)u$ is bounded in R_1.

which show that

$$B = -A - AB - \Pi_1,\ -B = A + BA + \Pi_0 \implies B = -A + A^2 + ABA - \Pi_1 + A\Pi_0. \tag{3.26}$$ [1.2.2000.329]

All terms here are in $\Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$; for ABA this follows from Proposition 3.3.

It remains to show the existence of the finite rank perturbation A'. This is equivalent to the vanishing of the index, that is

$$\operatorname{Ind}(\operatorname{Id} + A) = \dim \operatorname{Nul}(\operatorname{Id} + A) - \dim \operatorname{Nul}(\operatorname{Id} + A^*) = 0. \tag{3.27}$$ [1.2.2000.323]

Indeed, let f_j and g_j, $j = 1, \dots, N$, be respective bases of the two finite dimensional spaces $\operatorname{Nul}(\operatorname{Id} + A)$ and $\operatorname{Nul}(\operatorname{Id} + A^*)$. Then

$$A' = \sum_{j=1}^{N} g_j(x)\overline{f_j(y)} \tag{3.28}$$ [1.2.2000.324]

is an isomorphism of N_0 onto N_1 which vanishes on R_0. Thus $\operatorname{Id} + A + sA'$ is the direct sum of $\operatorname{Id} + A$ as an operator from R_0 to R_1 and sA' as an operator from N_0 to N_1, invertible when $s \neq 0$.

There is a very simple proof[2] of the equality (3.27) if we use the trace functional discussed in Section 3.14 below; this however is logically suspect as we use (although not crucially) approximation by finite rank operators in the discussion of the trace and this in turn might appear to use the present result via the discussion of ellipticity and the harmonic oscillator. Even though this is not really the case we give a clearly independent, but less elegant proof.

Consider the one-parameter family of operators $\operatorname{Id} + tA$, $A \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$. We shall see that the index, the difference in dimension between $\operatorname{Nul}(\operatorname{Id} + tA)$ and $\operatorname{Nul}(\operatorname{Id} + tA^*)$ is locally constant. To see this it is enough to consider a general A near the point $t = 1$. Consider the pieces of A with respect to the decompositions $L^2(\mathbb{R}^n) = N_i \oplus R_i$, $i = 0, 1$, of domain and range. Thus A is the sum of four terms which we write as a 2×2 matrix

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}.$$

Since $\operatorname{Id} + A$ has only one term in such a decomposition, $\tilde{A}$ in the lower right, the solution of the equation $(\operatorname{Id} + tA)u = f$ can be written

$$(t-1)A_{00}u_0 + (t-1)A_{01}u_\perp = f_1,\ (t-1)A_{10}u_0 + (A' + (t-1)A_{11})u_\perp = f_\perp \tag{3.29}$$ [1.2.2000.325]

[2]Namely the trace of a finite rank projection, such as either Π_0 or Π_1, is its rank, hence the dimension of the space onto which it projects. From the identity satisfied by the generalized inverse we see that

$$\operatorname{Ind}(\operatorname{Id} + A) = \operatorname{Tr}(\Pi_0) - \operatorname{Tr}(\Pi_1) = \operatorname{Tr}\left((\operatorname{Id} + B)(\operatorname{Id} + A) - (\operatorname{Id} + A)(\operatorname{Id} + B)\right) = \operatorname{Tr}([B, A]) = 0$$

from the basic property of the trace.

Since $\tilde{A}$ is invertible, for $t-1$ small enough the second equation can be solved uniquely for $u_\perp$. Inserted into the first equation this gives

$$
\begin{aligned}
(3.30)\quad G(t)u_0 &= f_1 + H(t)f_\perp,\\
G(t) &= (t-1)A_{00} - (t-1)^2 A_{01}(A' + (t-1)A_{11})^{-1}A_{10},\\
H(t) &= -(t-1)A_{01}(A' + (t-1)A_{11})^{-1}.
\end{aligned}
$$
[1.2.2000.326]

The null space is therefore isomorphic to the null space of $G(t)$ and a complement to the range is isomorphic to a complement to the range of $G(t)$. Since $G(t)$ is a finite rank operator acting from N_0 to N_1 the difference of these dimension is constant in t, namely equal to $\dim N_0 - \dim N_1$, near $t=1$ where it is defined.

This argument can be applied to tA so the index is actually constant in $t \in [0,1]$ and since it certainly vanishes at $t=0$ it vanishes for all t. In fact, as we shall note below, $\mathrm{Id} + tA$ is invertible outside a discrete set of $t \in \mathbb{C}$. □

[1.2.2000.330]

COROLLARY 3.1. *If* $\mathrm{Id}+A$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *is injective or surjective on* $L^2(\mathbb{R}^n)$, *in particular if it is invertible as a bounded operator, then it has an inverse in the ring* $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

[1.2.2000.333]

COROLLARY 3.2. *If* $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *then as an operator on* $\mathcal{S}(\mathbb{R}^n)$ *or* $\mathcal{S}'(\mathbb{R}^n)$, $\mathrm{Id}+A$ *is Fredholm in the sense that its null space is finite dimensional and its range is closed with a finite dimensional complement.*

PROOF. This follows from the existence of the generalized inverse of the form $\mathrm{Id}+B$, $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. □

3.5. Exponential and logarithm

[1.2.2000.350]

PROPOSITION 3.5. *The exponential*

$$
\exp(A) = \sum_j \frac{1}{j!}A^j : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.31}
$$
[1.2.2000.351]

is a globally defined, entire, function with range containing a neighbourhood of the identity and with inverse on such a neighbourhood given by the analytic function

$$
\log(\mathrm{Id}+A) = \sum_j \frac{(-1)^j}{j}A^j,\ A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \|A\|_{L^2} < 1 \tag{3.32}
$$
[1.2.2000.352]

3.6. Fredholm property[S.Fredholm.property]

An element $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is said to be elliptic (of order m in the isotropic calculus) if its left-reduced symbol is elliptic in $S^m_\infty(\mathbb{R}^{2n})$.

[21.2.1998.106]

THEOREM 3.3. *Each elliptic element* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *has a two-sided parametrix* $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *in the sense that*

$$
A \circ B - \mathrm{Id},\ B \circ A - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.33}
$$
[1.2.2000.365]

and it follows that any $u \in \mathcal{S}'(\mathbb{R}^n)$ *satisfying* $Au \in \mathcal{S}(\mathbb{R}^n)$ *is an element of* $\mathcal{S}(\mathbb{R}^n)$; *if* $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ *is elliptic then its parametrix is in* $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$.

PROOF. This is just the inductive argument used to prove Lemma 2.7. Nevertheless we repeat it here.

The existence of a right inverse for $\sigma_k(A)$ means that the equation $\sigma_k(A)c = d$ always has a solution $c \in \mathcal{A}_{j-k}$ for given $d \in \mathcal{A}_j$, namely $c = bd$. This in turn means that given $C_j \in \Psi^{j}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ there always exists $B_j \in \Psi^{j-k}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ such that $AB_j - D_j \in \Psi^{j-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Choosing $B_0 \in \Psi^{-k}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ to have $\sigma_{-k}(B_0) = b$ we can define $C_1 = \mathrm{Id} - AB_0 \in \Psi^{-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then, proceeding inductively we may assume that B_j for $j < l$ have been chosen such that $A(B_0 + \cdots + B_{l-1}) - \mathrm{Id} = -C_l \in \Psi^{-l}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then using the solvability we may choose B_l so that $AB_l - C_l = -C_{l+1} \in \Psi^{-l-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ which completes the induction, since $A(B_0 + \cdots + B_l) - \mathrm{Id} = AB_l - C_l = -C_{l+1}$. Finally by the asymptotic completeness we may choose $B \sim B_0 + B_1 + \dots$ which is a right parametrix.

The argument showing the existence of a left parametrix for a left-elliptic operator is completely analogous. □

Combining the earlier symbolic discussion and these analytic results we can see that elliptic operators in these calculi are Fredholm.

[21.3.1998.169]

PROPOSITION 3.6. *If $A \in \Psi^{m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ (resp. $A \in \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$) is elliptic then it has a generalized inverse $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ (resp. $B \in \Psi^{-l,-m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$) satisfying*

$$AB - \mathrm{Id} = \Pi_1,\ BA - \mathrm{Id} = \Pi_0 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)^{[21.3.1998.170]} \tag{3.34}$$

where Π_1 and Π_0 are the finite rank orthogonal (in $L^2(\mathbb{R}^n)$) projections onto the null spaces of A^* and A.

PROOF. In the case of an elliptic isotropic operator or order m we know that it has a parametrix $B' \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ modulo $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus

$$\begin{aligned} AB' &= \mathrm{Id} - E_R,\ E_R \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\\ B'A &= \mathrm{Id} - E_L,\ E_L \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \end{aligned}$$

Using Proposition 3.4 it follows that the null space of A is contained in the null space of $B'A = \mathrm{Id} - E_L$, hence is finite dimensional. Similarly, the range of A contains the range of $AB' = \mathrm{Id} - E_R$ so is closed with a finite codimensional complement. Defining B as the linear map which vanishes on $\mathrm{Nul}(A^*)$, and inverts A on $\mathrm{Ran}(A)$ with values in $\mathrm{Ran}(A^*) = \mathrm{Nul}(A)^\perp$ gives (3.34). Furthermore these identities show that $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ since applying B' gives

$$\begin{aligned} B - E_L B = B'AB = B' - B'\Pi_1,\ B - BE_R = BAB' = B' - \Pi_0 B' \Longrightarrow\\ B = B' - B'\Pi_1 + E_L B' + E_L B E_R - E_L \Pi_0 B' \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n).^{[1.2.2000.331]} \end{aligned} \tag{3.35}$$

The proof in the scattering case is essentially the same. □

[1.2.2000.332]

COROLLARY 3.3. *If $A \in \Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n)$ is elliptic then its generalized inverse lies in $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ and similarly if $A \in \Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)$ is elliptic then its generalized inverse lies in $\Psi^{-l,-m}_{\mathrm{sc}}(\mathbb{R}^n)$.*

3.7. The harmonic oscillator [S.Harmonic.oscillator]

The harmonic oscillator is the differential operator on $\mathbb{R}^n$

$$H = \sum_{j=1}^{n} (D_j^2 + x_j^2) = \Delta + |x|^2 .$$

This is an elliptic element of $\Psi^2_{\mathrm{iso}}(\mathbb{R}^n)$. The main immediate interest is in the spectral decomposition of H. The ellipticity of $H - \lambda$, $\lambda \in \mathbb{C}$, shows that

$$(H-\lambda)u = 0,\ u \in \mathcal{S}'(\mathbb{R}^n) \Longrightarrow u \in \mathcal{S}(\mathbb{R}^n)\,. \tag{3.36}$$ [eq:HO.1]

Since H is (formally) self-adjoint, i.e., $H^* = H$, there are no non-trivial tempered solutions of $(H-\lambda)u = 0$, $\lambda \notin \mathbb{R}$. Indeed if $(H-\lambda)u = 0$,

$$0 = \langle Hu, u\rangle - \langle u, Hu\rangle = (\lambda - \overline{\lambda})\langle u, u\rangle \Longrightarrow u = 0. \tag{3.37}$$ [eq:HO.2]

As we shall see below in more generality, the spectrum of H is a discrete subset of $\mathbb{R}$. In this case we can compute it explicitly.

The direct computation of eigenvalues and eigenfunctions is based on the properties of the creation and annihilation operators

$$C_j = D_j + ix_j, \quad C_j^* = A_j = D_j - ix_j,\ j = 1, \dots, n. \tag{3.38}$$ [eq:HO.3]

These satisfy the elementary identities

$$\begin{gathered} = [A_j, A_k] = 0,\ [A_j, C_k] = 2\delta_{jk},\ j, k = 1, \dots, n \\ H = \sum_{j=1}^{n} C_j A_j + n, \quad [C_j, H] = -2C_j, \quad [A_j, H] = 2A_j\,. \end{gathered} \tag{3.39}$$ [eq:HO.4]

Now, if λ is an eigenvalue, $Hu = \lambda u$, then

$$\begin{aligned} H(C_j u) &= C_j(Hu + 2u) = (\lambda+2)C_j u, \\ H(A_j u) &= A_j(Hu - 2u) = (\lambda - 2)A_j u. \end{aligned} \tag{3.40}$$ [eq:HO.5]

[prop:HO.6]

PROPOSITION 3.7. *The eigenvalues of H are*

$$\sigma(H) = \{n, n+2, n+4, \dots\}. \tag{3.41}$$ [eq:HO.7]

PROOF. We already know that eigenvalues must be real and from the decomposition of H in (3.40) it follows that, for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle Hu, u\rangle = \sum_j \|A_j u\|^2 + n\|u\|^2\,. \tag{3.42}$$ [eq:HO.9]

Thus if $\lambda \in \sigma(H)$ is an eigenvalue then $\lambda \geq n$.

By direct computation we see that n is an eigenvalue with a 1-dimensional eigenspace. Indeed, from (3.42), $Hu = nu$ iff $A_j u = 0$ for $j = 1, \dots, n$. In each variable separately

$$A_j u(x_j) = 0 \Leftrightarrow u(x_j) = c_1 \exp\left(-\frac{x_j^2}{2}\right).$$

Thus the only tempered solutions of $A_j u = 0$, $i = 1, \dots, n$ are the constant multiples of

$$u_0 = \exp\left(-\frac{|x|^2}{2}\right), \tag{3.43}$$ [eq:HO.10]

which is often called the *ground state.*

Now, if λ is an eigenvalue with eigenfunction $u \in \mathcal{S}(\mathbb{R}^n)$ it follows from (3.40) that $\lambda - 2$ is an eigenvalue with eigenfunction $A_j u$. Since all the $A_j u$ cannot vanish unless u is the ground state, it follows that the eigenvalues are contained in the set in (3.41). We can use the same argument to show that if u is an eigenfunction with eigenvalue λ then $C_j u$ is an eigenfunction with eigenvalue $\lambda + 2$. Moreover, $C_j u \equiv 0$ would imply $u \equiv 0$ since $C_j v = 0$ has no non-trivial tempered solutions, the solution in each variable being $\exp(x_j^2/2)$. $\square$

Using the creation operators we can parameterize the eigenspaces quite explicitly.

[prop:HO.11]

PROPOSITION 3.8. *For each $k \in \mathbb{N}_0$ there is an isomorphism*

$$\{\textit{Polynomials, homogeneous of degree } k \textit{ on } \mathbb{R}^n\} \ni p \longmapsto p(C)\exp\left(-\frac{|x|^2}{2}\right) \in E_k \quad \text{[eq:HO.12]} \tag{3.44}$$

where E_k is the eigenspace of H with eigenvalue $n + 2k$.

PROOF. Notice that the C_j, $j = 1, \dots, n$ are commuting operators, so $p(C)$ is well-defined. By iteration from (3.40),

$$HC^\alpha u_0 = C^\alpha(H + 2|\alpha|)u_0 = (n + 2|\alpha|)C^\alpha u_0\,. \quad \text{[eq:HO.13]} \tag{3.45}$$

Thus (3.44) is a linear map into the eigenspace as indicated.

To see that (3.44) is an isomorphism consider the action of the annihilation operators. Again from (3.40)

$$|\beta| = |\alpha| \Longrightarrow A^\beta C^\alpha u_0 = \begin{cases} 0 & \beta \neq \alpha \quad \text{[eq:HO.14]} \\ 2^{|\alpha|}\alpha! u_0 & \beta = \alpha. \end{cases} \tag{3.46}$$

This allows us to recover the coefficients of p from $p(C)u_0$, so (3.44) is injective. Conversely if $v \in E_k \subset \mathcal{S}(\mathbb{R}^n)$ is orthogonal to all the $C^\alpha u_0$ then

$$\langle A^\alpha v, u_0\rangle = \langle v, C^\alpha u_0\rangle = 0 \;\forall\; |\alpha| = k\,. \quad \text{[eq:HO.15]} \tag{3.47}$$

From (3.40), the $A^\alpha v$ are all eigenfunctions of H with eigenvalue n, so (3.47) implies that $A^\alpha v = 0$ for all $|\alpha| = k$. Proceeding inductively in k we see that $A^{\alpha'} A_j v = 0$ for all $|\alpha'| = k - 1$ and $A_j v \in E_{k-1}$ implies $A_j v = 0$, $j = 1, \dots, n$. Since $v \in E_k$, $k > 0$, this implies $v = 0$ so Proposition 3.8 is proved. $\square$

Thus H has eigenspaces as described in (3.44). The same argument shows that for any integer p, positive or negative, the eigenvalues of H^p are precisely $(n+2k)^p$ with the same eigenspaces E_k. For $p < 0$, H^p is a compact operator on $L^2(\mathbb{R}^n)$; this is obvious for large negative p. For example, if $p \le -n - 1$ then

$$x_i^\beta D_j^\alpha H \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n),\; |\alpha| \le n + 1, |\beta| \le n + 1 \quad \text{[eq:HO.16]} \tag{3.48}$$

are all bounded on L^2. If $S \subset L^2(\mathbb{R}^n)$ is bounded this implies that $H^{-n-1}(S)$ is bounded in $\langle x\rangle^{n+1}C^1_\infty(\mathbb{R}^n)$, so compact in $\langle x\rangle^n C^0_\infty(\mathbb{R}^n)$ and hence in $L^2(\mathbb{R}^n)$. It is a general fact that for compact self-adjoint operators, such as H^{-n-2}, the eigenfunctions span $L^2(\mathbb{R}^n)$. We give a brief proof of this for the sake of 'completeness'.

[lem:HO.17]

LEMMA 3.4. *The eigenfunction of H, $u_\alpha = \pi^{-\frac{n}{4}}(2^{|\alpha|}\alpha!)^{-1/2}C^\alpha u_0$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.*

PROOF. Let $V \subset L^2(\mathbb{R}^n)$ be the closed subspace consisting of the orthocomplements of all the u_α's. Certainly H^{-n-2} acts on it as a compact self-adjoint operator. Since we have found all the eigenvalues of H, and hence of H^{-n-1}, it has no eigenvalue in V. We wish to conclude that $V = \{0\}$. Set

$$\tau = \|H^{-n-1}\|_V = \sup\{\|H^{-n-1}\varphi\|; \varphi \in V, \|\varphi\| = 1\}\,.$$

Then there is a weakly convergent sequence $\varphi_j \rightharpoonup \varphi$, $\|\varphi_j\| = 1$, so $\|\varphi\| \le 1$, with $\|H^{-n-1}\varphi_j\| \to \tau$. The compactness of H^{-n-2} allows a subsequence to be chosen such that $H^{-n-1}\varphi_j \to \psi$ in $L^2(\mathbb{R}^n)$. So, by the continuity of H^{-n-1}, $H^{-n-1}\varphi = \psi$ and $\|H^{-n-1}\varphi\| = \tau$, $\|\varphi\| = 1$. If $\varphi' \in V$, $\varphi' \perp \varphi$, $\|\varphi'\| = 1$ then

$$\begin{aligned}\tau^2 \ge \|H^{-n-2}\left(\frac{\varphi + t\varphi'}{\sqrt{1+t^2}}\right)\|^2 &= \tau^2 + 2t\langle H^{-2n-2}\varphi, \varphi'\rangle + 0(t^2)\\ &\Longrightarrow \langle H^{-2n-2}\varphi, \varphi'\rangle = 0 \Longrightarrow H^{-2n-2}\varphi = \tau^2\varphi\,.\end{aligned}$$

This contradicts the fact that H^{-2n-2} has no eigenvalues in V, so $V = \{0\}$ and the eigenbasis is complete. □

Thus, if $u \in L^2(\mathbb{R}^n)$ then

$$u = \sum_\alpha c_\alpha u_\alpha, \quad c_\alpha = \langle u, u_\alpha\rangle\,. \tag{3.49}$$
[eq:HO.18]

[lem:HO.19]

LEMMA 3.5. *If $u \in \mathcal{S}(\mathbb{R}^n)$ the convergence in (3.49) is rapid, i.e., $|C_\alpha| \le C_N(1+|\alpha|)^{-N}$ for all N and the series converges in $\mathcal{S}(\mathbb{R}^n)$.*

PROOF. Since $u \in \mathcal{S}(\mathbb{R}^n)$ implies $H^N u \in L^2(\mathbb{R}^n)$ we see that

$$C_N \ge |\langle H^N u, u_\alpha\rangle| = |\langle u, H^N u_\alpha\rangle| = (n + 2|\alpha|)^N |c_\alpha|\ \forall\ \alpha\,.$$

Furthermore, $2ix_j = C_j - A_j$ and $2D_j = C_j + A_j$ so the polynomial derivatives of the u_α can be estimated (using the Sobolev embedding theorem) by polynomials in α; this implies that the series converges in $\mathcal{S}(\mathbb{R}^n)$. □

[1.2.2000.407]

COROLLARY 3.4. *Finite rank elements are dense in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^{2n})$.*

PROOF. Consider the approximation (3.49) to the kernel A of an element of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ as an element of $\mathcal{S}(\mathbb{R}^{2n})$. In this case the ground state is

$$U_0 = \exp\left(-\frac{|x|^2}{2} - \frac{|y|^2}{2}\right) = \exp\left(-\frac{|x|^2}{2}\right)\exp\left(-\frac{|y|^2}{2}\right)$$

and so has rank one as an operator. The higher eigenfunctions

$$C^\alpha U_0 = Q_\alpha(x,y)U_0$$

are products of U_0 and a polynomial, so are also of finite rank. □

3.8. L^2 boundedness and compactness

Recall that $\Psi^0_{\infty-\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^0_\infty(\mathbb{R}^n)$ so, by Proposition 2.6, these operators are bounded on $L^2(\mathbb{R}^n)$. Using the same argument the bound on the L^2 norm can be related to the norm of the principal symbol and an $N \times N$ matrix.

[1.2.2000.357]

PROPOSITION 3.9. *If* $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ *has principal symbol*

$$a = \sigma_L(A)\big|_{\mathbb{S}^{2n-1}} \in \mathcal{C}^\infty(\mathbb{S}^{2n-1}; M(N,\mathbb{C})$$

then

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)} \|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n;\mathbb{C}^N)} \le \sup_{p\in\mathbb{S}^{2n-1}} \|a(p)\|.^{[1.2.2000.358]} \tag{3.50}$$

PROOF. It suffices to prove (3.50) for all single operators $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$. Indeed if $j_v(z) = zv$ is the linear map from $\mathbb{C}$ to $\mathbb{C}^N$ defined by $v \in \mathbb{C}^N$ then

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N))} = \sup_{\{v,w\in\mathbb{C}^N;\|v\|=\|w\|=1\}} \|j_w^* A j_v\|_{\mathcal{B}(L^2(\mathbb{R}))}.^{[1.2.2000.363]} \tag{3.51}$$

Since the symbol of $j_w^* A j_v$ is just $j_w^*\sigma(A)j_v$, (3.50) follows from the corresponding equality for a single operator:

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)} \|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n)} \le \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|,\ a = \sigma_L(A)\big|_{\mathbb{S}^{2n-1}}.^{[1.2.2000.364]} \tag{3.52}$$

The construction of the approximate square-root of $C - A^*A$ in Proposition 2.7 only depends on the existence of a positive smooth square-root for $C - |a|^2$, so can be carried out for any

$$C > \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|^2.^{[1.2.2000.359]} \tag{3.53}$$

Thus we conclude that with such a value of C

$$\|Au\|^2 \le C\|u\|^2 + \|\langle Gu,u\rangle| \ \forall\ u \in L^2(\mathbb{R}^n),$$

where $G \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Since G is an isotropic smoothing operator, for any $\delta > 0$ there is a finite dimensional subspace $W \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\|\langle Gu,u\rangle\| \le \delta\|u\|^2 \ \forall\ u \in W^\perp.^{[1.2.2000.360]} \tag{3.54}$$

Thus if we replace A by $A(\mathrm{Id} - \Pi_W) = A + E$ where E is a (finite rank) smoothing operator we see that

$$\|(A+E)u\|^2 \le (C+\delta)\|Gu\|^2 \ \forall\ u \in L^2(\mathbb{R}^n) \Longrightarrow \|(A+E)\| \le (C+\delta)^{\frac{1}{2}}.^{[1.2.2000.361]}$$

This proves half of the desired estimate (3.51), namely

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)} \|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n)} \le \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|.^{[1.2.2000.362]} \tag{3.55}$$

To prove the opposite inequality, leading to (3.55), it is enough to arrive at a contradiction by supposing to the contrary that there is some $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ satisfying the strict inequality

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R}^n)} < \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|.$$

From this it follows that we may choose $c > 0$ such that $c = |a(p)|^2$ for some $p \in \mathbb{S}^{2n-1}$ and yet $A' = A^*A - c$ has a bounded inverse, B. By making an arbitrariy small perturbation of the full symbol of A' we may assume that it vanishes identically near p. By (3.55) we may choose $G \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with arbitrariy small L^2 such that $\tilde{A} = A' + B$ has left symbol rapidly vanishing near p. When the norm of the perturbation is small enough, $\tilde{A}$ will still be invertible, with inverse $\tilde{B} \in \mathcal{B}(L^2(\mathbb{R}^n)$. Now choose an element $G \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with left symbol supported sufficiently near p, so that $G \circ \tilde{A} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ but yet the principal symbol of G should not vanish at p. Thus

$$G = G \circ \tilde{A} \circ \tilde{B} : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ G* = G = \tilde{B}^* \circ \tilde{A}^* \circ G^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

It follows that $G^*G : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is an isotropic smoothing operator. This is the desired contrdiction, since G, and hence G^*G, may be chosen to have non-vanishing principal symbol at p. Thus we have proved (3.55) and hence the Proposition. □

It is then easy to characterize the compact operators amongst the polyhomogeneous isotropic operators as those of negative.

[1.2.2000.366]

LEMMA 3.6. *If $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then A is compact as an operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ if and only if it has negative order.*

PROOF. The necessity of vanishing of the principal symbol for a compact operator follows from Proposition 3.9 and the sufficiency follows from the density of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ in $\Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ in the topology of $\Psi^{-\frac{1}{2}}_{\infty-\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ and hence in the topology of bounded operators. Thus such an operator is the norm limit of compact operators so itself is compact. □

Also as a consequence of Proposition 3.9 we can see the necessity of the assumption of ellipticity in Proposition 3.6.

[1.2.2000.367]

COROLLARY 3.5. *If $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then A is Fredholm as an operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ if and only if it is elliptic.*

3.9. Sobolev spaces

There is a natural collection of Sobolev spaces associated to the isotropic calculus, and another associated to the scattering calculus.

Note that a differential operator $P(x, D_x)$ on $\mathbb{R}^n$ is an isotropic pseudodifferential operator if and only if its coefficients are polynomials. The fundamental symmetry between coefficients and differentiation suggest that the isotropic Sobolev spaces of non-negative integral order be defined by

$$H^k_{\mathrm{iso}}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n) \text{ if } |\alpha| + |\beta| \le k\},\ k \in \mathbb{N}. \tag{3.56}$$ [1.2.2000.337]

The norms

$$\|u\|^2_{k,\mathrm{iso}} = \sum_{|\alpha|+|\beta|\le k} \int_{\mathbb{R}^n} |x^\alpha D^\beta_x u|^2\, dx \tag{3.57}$$ [1.2.2000.339]

turn these into Hilbert spaces. For negative integral orders we identify the isotropic Sobolev spaces with the duals of these spaces

$$H^k_{\mathrm{iso}}(\mathbb{R}^n) = (H^{-k}_{\mathrm{iso}}(\mathbb{R}^n))' \hookrightarrow \mathcal{S}'(\mathbb{R}^n),\ k \in -\mathbb{N}. \tag{3.58}$$ [1.2.2000.340]

The (continuous) injection into tempered distributions here arises from the density of the image of the inclusion $\mathcal{S}(\mathbb{R}^n) \longrightarrow H^k_{\text{iso}}(\mathbb{R}^n)$.
[1.2.2000.338]

LEMMA 3.7. *For any $k \in \mathbb{Z}$,*

$$(3.59)\quad \begin{aligned} H^k_{\text{iso}}(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n)\ \forall\ A \in \Psi^{-k}_{\text{iso}}\} \\ &= \{u \in \mathcal{S}'(\mathbb{R}^n); \exists\ A \in \Psi^{-k}_{\text{iso}} \textit{ elliptic and such that } Au \in L^2(\mathbb{R}^n)\}^{[1.2.2000.341]} \end{aligned}$$

and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\text{iso}}(\mathbb{R}^n)$ is dense for each $k \in \mathbb{Z}$.

PROOF. [3] For $k \in \mathbb{N}$, the functions $x^\alpha \xi^\beta$ for $|\alpha| + |\beta| = k$ are 'collectively elliptic' in the sense that

$$(3.60)\qquad q_k(x,\xi) = \sum_{|\alpha|+|\beta|=k} (x^\alpha \xi^\beta)^2 \geq c(|x|^2 + |\xi|^2)^k,\ c > 0.^{[1.2.2000.344]}$$

Thus $Q_k = \sum\limits_{|\alpha|+|\beta|\leq k} (D^\beta x^\alpha x^\alpha D^\beta) \in \Psi^{2k}_{\text{iso}}(\mathbb{R}^n)$, which has principal reduced symbol q_k, has a left parameterix $A_k \in \Psi^{-2k}_{\text{iso}}(\mathbb{R}^n)$. This gives the identity

$$(3.61)\quad \sum_{|\alpha|+|\beta|\leq k} R_{\alpha,\beta} x^\alpha D^\beta = A_k Q_k = \mathrm{Id} + E,\ \text{where}$$
$$R_{\alpha,\beta} = A_k D^\beta x^\alpha \in \Psi^{-2k+|\alpha|+|\beta|}_{\text{iso}}(\mathbb{R}^n),\ E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n).^{[1.2.2000.345]}$$

Thus if $A \in \Psi^k_{\text{iso}}(\mathbb{R}^n)$

$$Au = -AEu + \sum_{|\alpha|+|\beta|\leq k} AR_{\alpha,\beta} x^\alpha D^\beta u.$$

If $u \in H^k_{\text{iso}}(\mathbb{R}^n)$ then by definition $x^\alpha D^\beta u \in L^2(\mathbb{R}^n)$. By the boundedness of operators of order 0 on L^2, all terms on the right are in $L^2(\mathbb{R}^n)$ and we have shown the inclusion of $H^k_{\text{iso}}(\mathbb{R}^n)$ in the first space space on the right in (3.59). The converse is immediate, so this proves the first equality in (3.59) for $k > 0$. Certainly the third space in (3.59) contains in the second. The existence of elliptic parametrix B for the ellipic operator A proves the converse since any isotropic pseudodifferential operator of order A' of order k can be effectively factorized as

$$A' = A'(BA + E) = B'A + E',\ B' \in \Psi^0_{\infty-\text{iso}}(\mathbb{R}^n),\ E' \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n).$$

Thus, $Au \in L^2(\mathbb{R}^n)$ implies that $A'u \in L^2(\mathbb{R}^n)$.

It also follows from second identification that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^k_{\text{loc}}(\mathbb{R}^n)$. Thus, if $Au \in L^2(\mathbb{R}^n)$ and we choose $f_n \in \mathcal{S}(\mathbb{R}^n)$ with $f_n \to Au$ in $L^2(\mathbb{R}^n)$ then, with B a parametrix for A, $u'_n = Bf_n \to BAu = u + Eu$. Thus $u_n = u'_n - Eu \in \mathcal{S}(\mathbb{R}^n) \to u$ in $L^2(\mathbb{R}^n)$ and $Au_n \to u$ in $L^2(\mathbb{R}^n)$ proving the density.

The Riesz representation theorem shows that $v\mathcal{S}'(\mathbb{R}^n)$ is in the dual space, $H^{-k}_{\text{iso}}(\mathbb{R}^n)$, if and only if there exists $v' \in H^k_{\text{iso}}(\mathbb{R}^n)$ such that

$$(3.62)\quad v(u) = \langle u, v'\rangle_{k,\text{iso}} = \langle u, Q_{2k}v'\rangle_{L^2},\ \forall\ u \in \mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\text{iso}}(\mathbb{R}^n)$$
$$\text{with } Q_{2k} = \sum_{|\alpha|+|\beta|\leq k} D^\beta x^{2\alpha} D^\beta.^{[1.2.2000.343]}$$

[3] This is an essentially microlocal proof.

This shows that Q_{2k} is an isomorphism of $H^k_{\mathrm{iso}}(\mathbb{R}^n)$ onto $H^{-k}_{\mathrm{iso}}(\mathbb{R}^n)$ as subspaces of $\mathcal{S}'(\mathbb{R}^n)$. Notice that $Q_{2k} \in \Psi^{2k}_{\mathrm{iso}}(\mathbb{R}^n)$ is elliptic, self-adjoint and invertible, since it is strictly positive. This now gives the same identification (3.59) for $k < 0$.

The case $k = 0$ follows directly from the L^2 boundedness of operators of order 0 so the proof is complete. □

In view of this identification we define the isotropic Sobolev spaces or any real order the same way

$$H^s_{\mathrm{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n)\ \forall\ A \in \Psi^{-s}_{\mathrm{iso}}\right\},\ s \in \mathbb{R}.^{[1.2.2000.347]} \tag{3.63}$$

These are Hilbertable spaces, with the Hilbert norm being given by $\|Au\|_{L^2(\mathbb{R}^n}$ for any $A \in \Psi^s_{\mathrm{iso}}(\mathbb{R}^n)$ which is elliptic and invertible.

[1.2.2000.346]

PROPOSITION 3.10. *Any element $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, $m \in \mathbb{R}$, defines a bounded linear operator*

$$A : H^s_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{s-m}_{\mathrm{iso}}(\mathbb{R}^n),\ \forall\ s \in \mathbb{R}.^{[1.2.2000.348]} \tag{3.64}$$

This operator is Fredholm if and only if A is elliptic. For any $s \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^s_{\mathrm{iso}}(\mathbb{R}^n)$ is dense and $H^{-s}_{\mathrm{iso}}(\mathbb{R}^n)$ may be identified as the dual of $H^s_{\mathrm{iso}}(\mathbb{R}^n)$ with respect to the continuous extension of the L^2 pairing.

PROOF. A straightforward application of the calculus, with the exception of the necessity of ellipticity for an isotropic pseudodifferential operator to be Fredholm. This is discussed in the problems beginning at Problem 3.10. □

3.10. The group $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)^{[sec:TG]}$

By definition, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the set (if you want to be concrete you can think of them as operators on $L^2(\mathbb{R}^n)$) of invertible elements of the ring $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. If we identify this topologically with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then, as follows from Corollary 3.1, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is open. We will think of it as an infinite-dimensional manifold modeled, of course, on the linear space $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n})$. Since I have no desire to get too deeply into the general theory of such Fréchet manifolds I will keep the discussion as elementary as possible.

The dual space of $\mathcal{S}(\mathbb{R}^p)$ is $\mathcal{S}'(\mathbb{R}^p)$. If we want to think of $\mathcal{S}(\mathbb{R}^p)$ as a manifold we need to consider smooth functions and forms on it. In the finite-dimensional case, the exterior bundles are the antisymmetric parts of the tensor powers of the dual. Since we are in infinite dimensions the tensor power needs to be completed and the usual choice is the 'projective' tensor product. In our case this is something quite simple namely the k-fold completed tensor power of $\mathcal{S}'(\mathbb{R}^p)$ is just $\mathcal{S}'(\mathbb{R}^{kp})$. Thus we set

$$\begin{aligned}\Lambda^k\mathcal{S}(\mathbb{R}^p) = \{u \in \mathcal{S}'(\mathbb{R}^{kp});\ &\text{for any permutation}\\ &e,\ u(x_{e(1)}, \dots x_{e(h)}) = \mathrm{sgn}(e)u(x_1, \dots x_k)\}.^{[TG.1]}\end{aligned} \tag{3.65}$$

In view of this it is enough for us to consider smooth functions on open sets $F \subset \mathcal{S}(\mathbb{R}^p)$ with values in $\mathcal{S}'(\mathbb{R}^p)$ for general p. Thus

$$v : F \longrightarrow \mathcal{S}'(\mathbb{R}^p),\ F \subset \mathcal{S}(\mathbb{R}^n)\ \text{open}^{[TG.2]} \tag{3.66}$$

is continuously differentiable on F if there exists a continuous map

$$v' : F \longrightarrow \mathcal{S}'(\mathbb{R}^{n+p}) \text{ and each } u \in F \text{ has a neighbourhood } U \\ \text{such that for each } N \ \exists\ M \quad \text{with} \\ \|v(u+u') - v(u) - v'(u;u')\|_N \leq C\|u'\|^2_M,\ \forall\ u, u+u' \in U.$$

Then, as usual we define smoothness as infinite differentiability by iterating this definition. The smoothness of v in this sense certainly implies that if $f : X \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is smooth then $v \circ F$ is smooth.

Thus we define the notion of a smooth *form* on $F \subset \mathcal{S}(\mathbb{R}^n)$, an open set, as a smooth map

$$\alpha : F \to \Lambda^k \mathcal{S}(\mathbb{R}^p) \subset \mathcal{S}'(\mathbb{R}^{kp})\,. \tag{3.67}$$ [TG.3]

In particular we know what smooth forms are on $\mathcal{G}^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

The de Rham differential acts on forms as usual. If $v : F \to \mathbb{C}$ is a function then its differential at $f \in F$ is $dv : F \longrightarrow \mathcal{S}'(\mathbb{R}^n) = \Lambda^1\mathcal{S}(\mathbb{R}^n)$, just the derivative. As in the finite-dimensional case d extends to forms by enforcing the condition that $dv = 0$ for constant forms and the identity distribution over exterior products

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta\,. \tag{3.68}$$ [TG.5]

3.11. Representations[S.Representations]

In §1.9 the compactification of Euclidean space to a ball, or half-sphere, is described. We make the following definition, recalling that $\rho \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$ is a boundary defining function.

[28.2.1998.153]

DEFINITION 3.2. *The space of of 'Laurent functions' on the half-sphere is*

$$\mathcal{L}(\mathbb{S}^{n,+}) = \bigcup_{k \in \mathbb{N}_0} \rho^{-k}\mathcal{C}^\infty(\mathbb{S}^{n,+}), \\ \rho^{-k}\mathcal{C}^\infty(\mathbb{S}^{n,+}) = \{u \in \mathcal{C}^\infty(\text{int}(\mathbb{S}^{n,+})); \rho^k u \in \mathcal{C}^\infty(\mathbb{S}^{n,+}). \tag{3.69}$$ [28.2.1998.154]

More generally if $m \in \mathbb{R}$ we denote by $\rho^m\mathcal{C}^\infty(\mathbb{S}^{n,+})$ the space of functions which can be written as products $u = \rho^m v$, with $v \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$; again it can be identified with a subspace of the space of $\mathcal{C}^\infty$ functions on the open half-sphere.

[28.2.1998.155]

PROPOSITION 3.11. *The compactification map* (1.94) *extends from* (1.96) *to give, for each $m \in \mathbb{R}$, an identification of $\rho^{-m}\mathcal{C}^\infty(\mathbb{S}^{n,+})$ and $S^m_{\text{cl}}(\mathbb{R}^n)$.*

Thus, the fact that the $\Psi^{\mathbb{Z}}_{\text{iso,cl}}(\mathbb{R}^n)$ form an order-filtered $*$-algebra means that $\rho^{\mathbb{Z}}\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ has a non-commutative product defined on it, with $\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ a subalgebra, using the left symbol isomorphism, followed by compactification.

3.12. Symplectic invariance of the isotropic product[S.Symplectic.invariance]

The composition law for the isotropic calculus, and in particular for it smoothing part, is derived from its identification as a subalgebra of the (weighted) spaces of pseudodifferential operator on $\mathbb{R}^n$. There is a much more invariant formulation of the product which puts into evidence more of the invariance properties.

Let W be a real symplectic vector space. Thus, W is a vector space equipped with a real, antisymmetic and non-degenerate bilinear form

$$\begin{aligned}(3.70)\quad \omega : W \times W \longrightarrow \mathbb{R},\ \omega(w_1, w_2) + \omega(w_2, w_1) = 0\ \forall\ w_1, w_2 \in W,\\ \omega(w_1, w) = 0\ \forall\ w \in W \Longrightarrow w_1 = 0.\end{aligned}$$ [23.3.1998.174]

A Lagrangian subspace of W is a vector space $V \subset W$ such that ω vanishes when restricted to V and such that $2\dim V = \dim W$.

[23.3.1998.175]

Lemma 3.8. *Every symplectic vector space has a Lagrangian subspace and for any choice of Lagrangian subspace U_1 there is a second Lagrangian subspace U_2 such that $W = U_1 \oplus U_2$ is a* Lagrangian decomposition.

Proof. First we show that there is a Lagrangian subspace. If $\dim W > 0$ then the antisymmetry of ω shows that any 1-dimensional vector subspace is *isotropic*, that is ω vanishes when restricted to it. Let V be a maximal isotropic subspace, that is an isotropic subspace of maximal dimension amongst isotropic subspaces. Let U be a complement to V in W. Then

$$(3.71)\qquad \omega : V \times U \longrightarrow \mathbb{R}$$ [23.3.1998.176]

is a non-degenerate pairing. Indeed $u \in U$ and $\omega(v, u) = 0$ for all $v \in V$ then $V + \mathbb{R}\{u\}$ is also isotropic, so $u = 0$ by the assumed maximality. Similarly if $v \in V$ and $\omega(v, u) = 0$ for all $u \in U$ then, recalling that ω vanishes on V, $\omega(v, w) = 0$ for all $w \in W$ so $v = 0$. The pairing (3.71) therefore identifies U with V', the dual of V. In particular $\dim w = 2\dim V$.

Now, choose any Lagrangian subspace U_1. We proceed to show that there is a complementary Lagrangian subspace. Certainly there is a 1-dimensional subspace which does not meet U_1. Let V be an isotropic subspace which does not meet U_1 and is of maximal dimension amongst such subspaces. Suppose that $\dim V < \dim U_1$. Choose $w \in W$ with $w \notin V \oplus U_1$. Then $V \ni v \longrightarrow \omega(w, v)$ is a linear functional on U_1. Since U_1 can be completed to a complement, any such linear functional can be written $\omega(u_1, v)$ for some $u_1 \in U_1$. It follows that $\omega(w - u_1, v) = 0$ for all $v \in V$. Thus $V \oplus \mathbb{R}\{w - u_1\}$ a non-trivial isotropic extension of V, contradicting the assumed maximality. Thus $V = U_2$ is a complement of U_1. □

Given such a Lagrangian decomposition of the symplectic vector space W, let $X_1, \dots X_n$ be a basis for the dual of U_1, and let $\Xi_1, \dots, \Xi_n$ be the dual basis, of U_1 itself. The pairing (3.71) with $U = U_1$ and $V = U_2$ identifies $U_2 = U_1'$ so the Ξ_i can also be regarded as a basis of the dual of U_2. Thus $X_1 \dots X_n, \Xi_1, \dots, \Xi_n$ gives a basis of $W' = U_1' \oplus U_2'$. The symplectic form can then be written

$$(3.72)\qquad \omega(w_1, w_2) = \sum_{i=1}^{n} (\Xi_i(w_1) X_i(w_2) - \Xi_i(w_2) X_i(w_1)).$$ [23.3.1998.178]

This is the *Darboux* form of ω. If the X_i, Ξ_i are thought of as linear functions x_i, ξ_i on W now considered as a manifold then these are *Darboux coordinates* in which(3.72) becomes

$$(3.73)\qquad \omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i.$$ [23.3.1998.179]

The symplectic form ω defines a volume form on W, namely the n-fold wedge product ω^n. In Darboux coordinates this is just, up to sign, the Lebesgue form $d\xi dx$.

[23.3.1998.180]

PROPOSITION 3.12. *On any symplectic vector space, W, the bilinear map on $\mathcal{S}(W)$,*

$$a\#b(w) = (2\pi)^{-2n}\int_{W^2} e^{i\omega(w_1,w_2)}a(w+w_1)b(w+w_2)\omega^n(w_1)\omega^2(w_2),\ \dim W = 2n \tag{3.74}$$ [23.3.1998.181]

defines an associative product isomorphic to the composition of $\Psi^{-\infty}_{\text{iso}}(U_1)$ for any Lagrangian decomposition $W = U_1 \oplus U_2$.

[23.3.1998.182]

COROLLARY 3.6. *Extended by continuity in the symbol space* (3.74) *defines a filtered product on $S^\infty(W)$ which is isomorphic to the isotropic algebra on $\mathbb{R}^{2n}$ and is invariant under symplectic linear transformation of W.*

PROOF. Written in the form (3.74) the symplectic invariance is immediate. That is, if F is a linear transformation of W which preserves the symplectic form, $\omega(Fw_1, Fw_2) = \omega(w_1, w_2)$ then

$$F^*(a\#b) = (F^*a)\#(F^*b)\ \forall\ a, b \in \mathcal{S}(W). \tag{3.75}$$ [23.3.1998.183]

The same result holds for general symbols once the continuity is established.

Let us start from the Weyl quatization of the isotropic algebra. As usual for computations we may assume that the amplitudes are of order $-\infty$. Thus, $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ may be written

$$Au(x) = \int A(x,y)u(y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(\frac{1}{2}(x+y),\xi)u(y)dyd\xi. \tag{3.76}$$ [23.3.1998.184]

Both the kernel $A(x, y)$ and the amplitude $a(x, \xi)$ are elements of $\mathcal{S}(\mathbb{R}^{2n})$. The relationship (3.76) and its inverse may be written

$$\begin{aligned} A(s+\frac{t}{2}, s-\frac{t}{2}) &= (2\pi)^{-n}\int e^{it\cdot\xi}a(s,\xi)d\xi, \\ a(x,\xi) &= \int e^{-it\cdot\xi}A(x+\frac{t}{2}, x-\frac{t}{2})dt. \end{aligned} \tag{3.77}$$ [23.3.1998.185]

If A has Weyl symbol a and B has Weyl symbol b let c be the Weyl symbol of the composite $A \circ B$. Using (3.77) and (3.76)

$$\begin{gathered} c(s,\zeta) = \int e^{-it\cdot\zeta}A(s+\frac{t}{2}, z)B(z, s-\frac{t}{2})dt \\ = (2\pi)^{-2n}\int\int\int dt\,dz\,d\xi\,d\eta e^{i\Phi}a(\frac{s}{2}+\frac{t}{4}+\frac{z}{2},\xi)a(\frac{z}{2}+\frac{s}{2}-\frac{t}{4},\eta) \\ \text{where } \Phi = -t\cdot\zeta + (s+\frac{t}{2}-z)\cdot\xi + (z-s+\frac{t}{2})\cdot\eta. \end{gathered}$$

Changing variables of integration to $X = \frac{z}{2} + \frac{t}{4} - \frac{s}{2}$, $Y = \frac{z}{2} - \frac{t}{4} - \frac{s}{2}$, $\Xi = \xi - \zeta$ and $H = \eta - \zeta$ this becomes

$$c(s, \zeta) = (2\pi)^{-2n} 4^n \int\int\int dY\, dX\, d\Xi\, dH \qquad e^{2i(X \cdot H - Y \cdot \Xi)} a(X + s, \Xi + \zeta) a(Y + s, H + \zeta).$$

This reduces to (3.74), written out in Darboux coordinates, after the change of variable $H' = 2H$, $\Xi' = 2\Xi$ and $\zeta' = 2\zeta$. Thus the precise isomorphism with the product in Weyl form is given by

$$A(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\omega(\frac{1}{2}(x + y), 2\xi) u(y) dy d\xi \tag{3.78}$$ [23.3.1998.186]

so that composition of kernels reduces to (3.74). □

Discuss metaplectic group here.

3.13. Complex order[S.Complex.order]

The identification of polyhomogeneous symbols of order zero on $\mathbb{R}^{2n}$ with the smooth functions on the radial compactification allows us to define the isotropic operators of a given complex order $z \in \mathbb{C}$. Namely, we use the left quantization map to identify

$$\Psi^z_{\mathrm{iso}}(\mathbb{R}^n) = \rho^{-z} \mathcal{C}^\infty(\mathbb{S}^{2n,1}) \subset \Psi^{\Re z}_{\infty-\mathrm{iso}}(\mathbb{R}^n). \tag{3.79}$$ [1.2.2000.334]

Here, $\rho \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$ is a boundary defining function. Any other boundary defining function is of the form $a\rho$ with $0 < a \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$. It follows that the definition is independent of the choice of ρ since $a^z \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$ for any $z \in \mathbb{Z}$.

In fact it is even more useful to consider holomorphic families. Thus if $\Omega \subset \mathbb{C}$ is an open set and $h : \Omega \longrightarrow \mathbb{C}$ is holomorphic then we may consider holomorphic families of order h as elements of

$$\Psi^{h(z)}_{\mathrm{iso}}(\mathbb{R}^{2n}) = \big\{A : \Omega \longrightarrow \Psi^\infty_{\infty-\mathrm{iso}}(\mathbb{R}^{2n}); \qquad \Omega \ni z \longmapsto \rho^{h(z)} A(z) \in \mathcal{C}^\infty(\mathbb{S}^{2n,1}) \text{ is holomorphic.}\big\} \tag{3.80}$$ [1.2.2000.335]

Note that a map from $\Omega \subset \mathbb{C}$ into $\mathcal{C}^\infty(\mathbb{S}^{2n1,})$ is said to be holomorphic it is defines an element of $\mathcal{C}^\infty(\Omega \times \mathbb{S}^{2n,1})$ which satisies the Cauchy-Riemann equation in the first variable.

[1.2.2000.336]

PROPOSITION 3.13. *If h and g are holomorphic functions on an open set $\Omega \subset \mathbb{C}$ and $A(z)$, $B(z)$ are holomorphic familes of isotropic operators of orders $h(z)$ and $g(z)$ then the composite family $A(z) \circ B(z)$ is holomorphic of order $h(z) + g(z)$.*

PROOF. It suffices to consider an arbitrary open subset $\Omega' \subset \Omega$ with compact closure inside Ω. Then h and g have bounded real parts, so $A(z)$, $B(z) \in \Psi^M_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$ for $z \in \Omega'$ for some fixed M. It follows that the composite $A(z) \circ B(z) \in \Psi^{2M}_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$. The symbol is given by the usual formula. Furthermore

□

3.14. Traces on the residual algebra[S.Traces.residual]

The algebras we are studying are topological algebras, so it makes sense to consider continuous linear functionals on them. The most important of these is the *trace*. To remind you what it is we consider first its properties for matrix algebras.

Let $M(N;\mathbb{C})$ denote the algebra of $N\times N$ complex matrices. We can simply define

$$\mathrm{Tr}: M(N;\mathbb{C}) \to \mathbb{C}, \quad \mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii} \tag{3.81}$$ [eq:1]

as the sum of the diagonal entries. The fundamental property of this functional is that

$$\mathrm{Tr}([A,B]) = 0 \ \forall\ A,B\in M(N;\mathbb{C})\,. \tag{3.82}$$ [eq:2]

To check this it is only necessary to write down the definition of the composition in the algebra. Thus

$$(AB)_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}\,.$$

It follows that

$$\begin{aligned}\mathrm{Tr}(AB) &= \sum_{i=1}^{N}(AB)_{ii} = \sum_{i,k=1}^{N} A_{ik}B_{ki}\\ &= \sum_{k=1}^{N}\sum_{i=1}^{N} B_{ki}A_{ik} = \sum_{k=1}^{N}(BA)_{kk} = \mathrm{Tr}(BA)\end{aligned}$$

which is just (3.82).

Of course any multiple of Tr has the same property (3.82) but the normalization condition

$$\mathrm{Tr}(\mathrm{Id}) = N \tag{3.83}$$ [eq:3]

distinguishes it from its multiples. In fact (3.82) and (3.83) together distinguish $\mathrm{Tr}\in M(N;\mathbb{C})'$ as a point in the N^2 dimensional linear space which is the dual of $M(N;\mathbb{C})$.

[lem:trace]

Lemma 3.9. *If $F: M(N;\mathbb{C})\to\mathbb{C}$ is a linear functional satisfying* (3.82) *and $B\in M(N;\mathbb{C})$ is any matrix such that $F(B)\neq 0$ then $F(A) = \frac{F(B)}{\mathrm{Tr}(B)}\mathrm{Tr}(A)$.*

Proof. Consider the basis of $M(N;\mathbb{C})$ given by the elementary matrices E_{jk}, where E_{jk} has jk-th entry 1 and all others zero. Thus

$$E_{jk}E_{pq} = \delta_{kp}E_{jq}.$$

If $j\neq k$ it follows that

$$E_{jj}E_{jk} = E_{jk},\ E_{jk}E_{jj} = 0.$$

Thus

$$F([E_{jj},E_{jk}]) = F(E_{jk}) = 0 \text{ if } j\neq k.$$

On the other hand, for any i and j

$$E_{ji}E_{ij} = E_{jj},\ E_{ij}E_{ji} = E_{ii}$$

so

$$F(E_{jj}) = F(E_{11})\ \forall\ j.$$

Since the E_{jk} are a basis,

$$\begin{aligned} F(A) &= F(\sum_{j,k=1}^{N} A_{ij}E_{ij}) \\ &= \sum_{j,l=1}^{N} A_{jj}F(E_{ij}) \\ &= F(E_{11})\sum_{j=1}^{N} A_{jj} = F(E_{11})\operatorname{Tr}(A). \end{aligned}$$

This proves the lemma. $\square$

For the isotropic smoothing algebra we have a similar result.

[isotropictrace]

PROPOSITION 3.14. *If* $F : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathbb{C}$ *is a continuous linear functional satisfying*

$$F([A,B]) = 0\ \forall\ A, B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)^{[eq:4]} \tag{3.84}$$

then $F([A,B]) = 0$ *for all* $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *and* $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *and* F *is a constant multiple of the functional*

$$\operatorname{Tr}(A) = \int_{\mathbb{R}^n} A(x,x)\,dx.^{[eq:5]} \tag{3.85}$$

PROOF. Recall that $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is an ideal so $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ implies that AB, $BA \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and it follows that the equality $F(AB) = F(BA)$, or $F([A,B]) = 0$, is meaningful. To see that it holds we just use the continuity of F. We know that if $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then there is a sequence $B_n \to B$ in the topology of $\Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n)$ for some m. Since this implies $AB_n \to AB$, $B_nA \to BA$ in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ we see that

$$F([A,B]) = \lim_{n\to\infty} F([A,B_n]) = 0\,.$$

We use this identity to prove (3.85). Take $B = x_j$ or D_j, $j = 1, \dots, n$. Thus for any $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$

$$F([A,x_j]) = F([A,D_j]) = 0\,.$$

Now consider F as a distribution acting on the kernel $A \in \mathcal{S}(\mathbb{R}^{2n})$. Since the kernel of $[A,x_j]$ is $A(x,y)(y_j - x_j)$ and the kernel of (A,D_j) is $-(D_{y_j} + D_{x_j})A(x,y)$ we conclude that, as an element of $\mathcal{S}'(\mathbb{R}^{2n})$, F satisfies

$$(x_j - y_j)F(x,y) = 0,\ (D_{x_j} + D_{y_j})F(x,y) = 0.$$

If we make the linear change of variables to $p_i = \frac{x_i+y_i}{2}$, $q_i = x_i - y_i$ and set $\tilde{F}(p,q) = F(x,y)$ these conditions become

$$D_{q_i}\tilde{F} = 0,\ p_i\tilde{F} = 0,\ i = 1, \dots, N.$$

As we know from Lemmas 1.2 and 1.3, this implies that $\tilde{F} = c\delta(p)$ so

$$F(x, y) = c\delta(x - y)$$

as a distribution. Clearly $\delta(x - y)$ gives the functional Tr defined by (3.85), so the proposition is proved. □

We still need to justify the use of the same notation, Tr, for these two functionals. However, if $L \subset \mathcal{S}(\mathbb{R}^n)$ is any finite dimensional subspace we may choose an orthonal basis $\varphi_i \in L$, $i = 1, \dots, l$,

$$\int_{\mathbb{R}^n} |\varphi_i(x)|^2 \, dx = 0, \ \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x) \, dx = 0, \ i \neq j \, .$$

Then if a_{ij} is an $l \times l$ matrix,

$$A = \sum_{i,j=1}^{\ell} a_{ij}\varphi_i(x)\overline{\varphi_j(y)} \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \, .$$

From (3.85) we see that

$$\begin{aligned} \operatorname{Tr}(A) &= \sum_{ij} a_{ij} \operatorname{Tr}(\varphi_i\bar{\varphi}_j) \\ &= \sum_{ij} a_{ij} \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x) \, dx \\ &= \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}(a) \, . \end{aligned}$$

Thus the two notions of trace coincide. In any case this already follows, up to a constant, from the uniqueness in Lemma 3.9.

3.15. Fredholm determinant

For $N \times N$ matrices, the linear space of which we denote $M(N; \mathbb{C})$, the determinant is a multiplicative polynomial map

(3.86) $$\det : M(N; \mathbb{C}) \longrightarrow \mathbb{C}, \ \det(AB) = \det(A)\det(B), \ \det(\mathrm{Id}) = 1.$$ [1.2.2000.404]

It is not quite determined by these conditions, since $\det(A)^k$ also satisfies then. The fundamental property of the determinant is that it defines the group of invertible elements

(3.87) $$\mathrm{GL}(N, \mathbb{C}) = \{A \in M(N; \mathbb{C}); \det(A) \neq 0\}.$$ [1.2.2000.405]

A reminder of a direct definition is given in Problem 3.7.

The Fredholm determinant is an extension of this definition to a function on the ring $\mathrm{Id} + \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$, or even further to $\mathrm{Id} + \Psi^{-2n-\epsilon}_{\infty-\text{iso}}(\mathbb{R}^n)$ for $\epsilon > 0$. This can be done in several ways using the density of finite rank operators, as shown in Corollary 3.4. We proceed by generalizing the formula relating the determinant to the trace. Thus, for any smooth curve with values in $\mathrm{GL}(N; \mathbb{C})$ for any N,

(3.88) $$\frac{d}{ds}\det(A_s) = \det(A_s)\operatorname{tr}(A_s^{-1}\frac{A_s}{ds}).$$ [1.2.2000.408]

In particular if (3.86) is augmented by the normalization condition

(3.89) $$\frac{d}{ds}\det(\mathrm{Id} + sA)\big|_{s=0} = \operatorname{tr}(A) \ \forall \ A \in M(N; \mathbb{C})$$ [iml.1] then it is determined.

A branch of the logarithm can be introduced along any curve, smoothly in the parameter, and then (3.88) can be rewritten

$$d \log \det(A) = \operatorname{tr}(A^{-1} dA). \tag{3.90}$$ [1.2.2000.409]

Here $\mathrm{GL}(N;\mathbb{C})$ is regarded as a subset of the linear space $M(N;\mathbb{C})$ and dA is the canonical identification, at the point A, of the tangent space to $M(N,\mathbb{C})$ with $M(N,\mathbb{C})$ itself. This just arises from the fact that $M(N,\mathbb{C})$ is a linear space. Thus $dA(\frac{d}{ds}(A+sB)\big|_{s=0} = B$. This allows the expression on the right in (3.90) to be interpreted as a smooth 1-form on the manifold $\mathrm{GL}(N;\mathbb{C})$. Note that it is independent of the local choice of logarithm.

To define the Fredholm determinant we shall extend the 1-form

$$\alpha = \operatorname{Tr}(A^{-1} dA) \tag{3.91}$$ [iml.2]

to the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \hookrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Here dA has essentially the same meaning as before, given that Id is fixed. Thus at any point $A = \mathrm{Id} + B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ it is the identification of the tangent space with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ using the linear structure:

$$dA(\frac{d}{ds}(\mathrm{Id} + B + sE)\big|_{s=0}) = E,\ E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$ [iml.3]

Since dA takes values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, the trace functional in (3.91) is well defined.

The 1-form α is closed. In the finite-dimensional case this follows from (3.90). For (3.91) we can compute directly. Since $d(dA) = 0$, essentially by definition, and

$$dA^{-1} = -A^{-1} dA A^{-1} \tag{3.92}$$ [iml.4]

we see that

$$d\alpha = -\operatorname{Tr}(A^{-1}(dA)A^{-1}(dA)) = 0. \tag{3.93}$$ [iml.5]

Here we have used the trace identity, and the antisymmetry of the implicit wedge product in (3.93), to conlcude that $d\alpha = 0$. For a more detailed discussion of this point see Problem 3.8.

From the fact that $d\alpha = 0$ we can be confident that there is, locally near any point of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, a function f such that $df = \alpha$; then we will define the Fredholm determinant by $\det_{\mathrm{Fr}}(A) = \exp(f)$. To define $\det_{\mathrm{Fr}}$ globally we need to see that this is well defined.

[iml.9]

Lemma 3.10. *For any smooth closed curve $\gamma : \mathbb{S}^1 \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the integral*

$$\int_\gamma \alpha = \int_{\mathbb{S}^1} \gamma^* \alpha \in 2\pi i \mathbb{Z}. \tag{3.94}$$ [iml.10]

That is, α defines an integral cohomology class, $[\frac{\alpha}{2\pi i}] \in H^1(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n);\mathbb{Z})$.

Proof. This is where we use the approximability by finite rank operators. If π_N is the orthogonal projection onto the span of the eigenspaces of the smallest N eigenvalues of the harmonic oscillator then we know from Section 3.7 that $\pi_N E \pi_N \to E$ in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for any element. In fact it follows that for the smooth curve that $\gamma(s) = \mathrm{Id} + E(s)$ and $E_N(s) = \pi_N E(s) \pi_N$ converges uniformly with all

s derivatives. Thus, for some N_0 and all $N > N_0$, $\mathrm{Id} + E_N(s)$ is a smooth curve in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and hence $\gamma_N(s) = \mathrm{Id}_N + E_N(s)$ is a smooth curve in $\mathrm{GL}(N;\mathbb{C})$. Clearly

$$(3.95) \qquad \int_{\gamma_N} \alpha \longrightarrow \int_\gamma \alpha \text{ as } N \to \infty, \quad [iml.11]$$

and for finite N it follows from the identity of the trace with the matrix trace (see Section 3.14) that $\int_N \gamma_N^* \alpha$ is the variation of $\arg\log\det(\gamma_N)$ around the curve. This gives (3.94). □

Now, once we have (3.94) and the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ we may define

$$(3.96) \qquad \det{}_{\mathrm{Fr}}(A) = \exp(\int_\gamma \alpha),\ \gamma : [0,1] \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \gamma(0) = \mathrm{Id},\ \gamma(1) = A. \quad [iml.12]$$

Indeed, Lemma 3.10 shows that this is independent of the path chosen from the identity to A. Notice that the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ follows from the connectedness of the $\mathrm{GL}(N,\mathbb{C})$ and the density argument above.

The same arguments and results apply to $G^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ using the fact that the trace functional extends continuously to $\Psi^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for any $\epsilon > 0$.

[iml.13]

PROPOSITION 3.15. *The Fredholm determinant, defined by* (3.96) *on* $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *(or* $G^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$ *for* $\epsilon > 0$) *and to be zero on the complement in* $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *(or* $\mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$) *is an entire function satisfying*

$$(3.97) \qquad \det{}_{\mathrm{Fr}}(AB) = \det{}_{\mathrm{Fr}}(A)\det{}_{\mathrm{Fr}}(B),\ A, B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \\ (\mathit{or}\ \mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)),\ \det{}_{\mathrm{Fr}}(\mathrm{Id}) = 1. \quad [iml.14]$$

PROOF. We start with the multiplicative property of $\det_{\mathrm{Fr}}$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus is $\gamma_1(s)$ is a smooth curve from Id to A_1 and $\gamma_2(s)$ is a smooth curve from Id to A_2 then $\gamma(s) = \gamma_1(s)\gamma_2(s)$ is a smooth curve from Id to A_1A_2. Consider the differential on this curve. Since

$$\frac{d(A_1(s)A_2(s))}{ds} = \frac{dA_1(s)}{ds}A_2(s) + A_1(s)\frac{dA_2(s)}{ds}$$

the 1-form becomes

$$(3.98) \qquad \gamma^*(s)\alpha(s) = \mathrm{Tr}(A_2(s)^{-1}\frac{dA_2(s)}{ds}) + \mathrm{Tr}(A_2(s)^{-1}A_1(s)^{-1}\frac{dA_2(s)}{ds}A_2(s)). \quad [iml.15]$$

In the second term on the right we can use the trace identity, since $\mathrm{Tr}(GA) = \mathrm{Tr}(AG)$ if $G \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ and $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus (3.98) becomes

$$\gamma^*(s)\alpha(s) = \gamma_1^*\alpha + \gamma_2^*\alpha.$$

Inserting this into the definition of $\det_{\mathrm{Fr}}$ gives (3.97) when both factors are in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Of course if either factor is not invertible, then so is the product and hence both $\det_{\mathrm{Fr}}(AB)$ and at least one of $\det_{\mathrm{Fr}}(A)$ and $\det_{\mathrm{Fr}}(B)$ vanishes. Thus (3.97) holds in general when $\det_{\mathrm{Fr}}$ is extended to be zero on the non-invertible elements.

Thus it remains to establish the smoothness. That $\det_{\mathrm{Fr}}(A)$ is smooth in any real parameters in which $A \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ depends, or indeed is holomorphic in holomorphic parameters, follows from the definition since α clearly depends smoothly, or holomorphically, on parameters. In fact the same follows if holomorphy is examined as a function of E, $A = \mathrm{Id} + E$, for $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus it is only smoothness across the non-invertibles that is at issue. To prove this we use the multiplicativity just established.

If $A = \mathrm{Id} + E$ is not invertible, $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then it has a generalized inverse $\mathrm{Id} + E'$ as in Proposition 3.6. Since A has index zero, we may actually replace E' by $E' + E''$, where E'' is an invertible linear map from the orthocomplement of the range of A to its null space. Then $\mathrm{Id} + E' + E'' \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $(\mathrm{Id} + E' + E'')A = \mathrm{Id} - \Pi_0$. To prove the smoothness of $\det_{\mathrm{Fr}}$ on a neighbourhood of A it is enough to prove the smoothness on a neighbourhood of $\mathrm{Id} - \Pi_0$ since $\mathrm{Id} + E' + E''$ maps a neighbourhood of the first to a neighbourhood of the second and $\det_{\mathrm{Fr}}$ is multiplicative. Thus consider $\det_{\mathrm{Fr}}$ on a set $\mathrm{Id} - \Pi_0 + E$ where E is near 0 in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, in particular we may assume that $\mathrm{Id} + E \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus

$$\det_{\mathrm{Fr}}(\mathrm{Id} + E - \Pi_0) = \det(\mathrm{Id} + E)\det(\mathrm{Id} - \Pi_0 + (G_E - \mathrm{Id})\Pi_0)^{[iml.16]}$$

were $G_E = (\mathrm{Id} + E)^{-1}$ depends holomorphically on E. Thus it suffices to prove the smoothness of $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H\Pi_0)$ where $H \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$

Consider the deformation $H_s = \Pi_0 H \Pi_0 + s(\mathrm{Id} - \Pi_0)H\Pi_0$, $s \in [0,1]$. If $\mathrm{Id} - \Pi_0 + H_s$ is invertible for one value of s it is invertible for all, since its range is always the range of $\mathrm{Id} - \Pi_0$ plus the range of $\Pi_0 H \Pi_0$. It follows that $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)$ is smooth in s; in fact it is constant. If the family is not invertible this follows immediately and if it is invertible then

$$\frac{d\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)}{ds} = \det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)\operatorname{Tr}\left((\mathrm{Id} - \Pi_0 + H_s)^{-1}(\mathrm{Id} - Pi_0)H\Pi_0)\right) = 0$$

since the argument of the trace is finite rank and off-diagonal with respect to the decomposition by Π_0.

Thus finally it is enough to consider the smoothness of $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + \Pi_0 H \Pi_0)$ as a function of $H \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Since this is just $\det(\Pi_0 H \Pi_0)$, interpreted as a finite rank map on the range of Π_0 the result follows from the finite dimensional case. □

[iml.18]

LEMMA 3.11. *If $A \in G^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ and $B \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then $ABA^{-1} \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and*

$$\det_{\mathrm{Fr}}(ABA^{-1}) = \det_{\mathrm{Fr}}(B).^{[iml.19]} \tag{3.99}$$

PROOF. If ABA^{-1} is not invertible then neither is B so both sides of (3.99) vanish. Thus we may assume that $B = \mathrm{Id} + E$ is invertible and let $\mathrm{Id} + E(s)$ be a smooth curve in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ connecting it to the identity. Consider the function

$$\det_{\mathrm{Fr}}(A(\mathrm{Id} + E(s))A^{-1}) = \det_{\mathrm{Fr}}(\mathrm{Id} + AE(s)A^{-1}).^{[iml.20]} \tag{3.100}$$

This is certainly smooth and non-vanishing and its logarithm has derivative

$$\operatorname{Tr}(A(\mathrm{Id} + E(s))^{-1}A^{-1}\frac{d\left(A(\mathrm{Id} + E(s)A^{-1}\right)}{ds}) = \operatorname{Tr}(A(\mathrm{Id} + E(s))^{-1}\frac{E(s)}{ds}A^{-1}) = \operatorname{Tr}((\mathrm{Id} + E(s))^{-1}\frac{E(s)}{ds}).$$

This is also the derivative of the logarithm of $\det_{\mathrm{Fr}}(\mathrm{Id} + E(s))$ so the result follows. □

3.16. Fredholm alternative[S.Fredholm.alternative]

Since we have shown that $\det_{\mathrm{Fr}} : \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is an entire function, we see that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the complement of a (singular) holomorphic hypersurface, namely the surface $\{\mathrm{Id} + E; \det_{\mathrm{Fr}}(\mathrm{Id} + E) = 0\}$. This has the following consequence, which is sometimes call the 'Fredholm alternative' and also part of 'analytic Fredholm theory'.

[iml.21]

LEMMA 3.12. *If $\Omega \subset \mathbb{C}$ is an open, connected set and $A : \Omega \longrightarrow \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic function then either $\mathrm{Id} + A(z)$ is invertible of all but a discrete subset of Ω and $(\mathrm{Id} + A(z))$ is meromorphic on Ω with all residues of finite rank, or else it is invertible at no point of Ω.*

PROOF. Of course the point here is that $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z))$ is a holomorphic function on Ω. Thus, either $\det_{\mathrm{Fr}}(A(z)) = 0$ is a discrete set, $D \subset \Omega$ or else $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z)) \equiv 0$ on Ω; this uses the connectedness of Ω. Since this corresponds exactly to the invertibility of $\mathrm{Id} + A(z)$ the main part of the lemma is proved. It remains only to show that, in the former case, $(\mathrm{Id} + A(z))^{-1}$ is meromorphic. Thus consider a point $p \in D$. Thus the claim is that near p

$$(\mathrm{Id} + A(z))^{-1} = \mathrm{Id} + E(z) + \sum_{j=1}^{N} z^{-j} E_j,\ E_j \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \text{ of finite rank} \tag{3.101}$$ [iml.22]

and where $E(z)$ is locally holomorphic with values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

If N is sufficiently large and Π_N is the projection onto the first N eigenspaces of the harmonic oscillator then $B(z) = \mathrm{Id} + E(z) - \Pi_N E(z) \Pi_N$ is invertible near p with the inverse being of the form $\mathrm{Id} + F(z)$ with $F(z)$ locally holomorphic. Now

$$\begin{aligned}(\mathrm{Id} + F(z))(\mathrm{Id} + E(z)) &= \mathrm{Id} + (\mathrm{Id} + F(z))\Pi_N E(z) \Pi_N \\ &= (\mathrm{Id} - \Pi_N) + \Pi_N M(z) \Pi_N + (\mathrm{Id} - \Pi_N) M'(z) \Pi_N.\end{aligned}$$

It follows that this is invertible if and only if $M(z)$ is invertible as a matrix on the range of Π_N. Since it must be invertible near, but not at, p, its inverse is a meromorphic matrix $K(z)$. It follows that the inverse of the product above can be written

$$\mathrm{Id} - \Pi_N + \Pi_N K(z) \Pi_N - (\mathrm{Id} - \Pi_N) M'(z) \Pi_N K(z) \Pi_N. \tag{3.102}$$ [iml.23]

This is meromorphic and has finite rank residues, so it follows that the same is true of $A(z)^{-1}$. □

This result for the smoothing operators, which really follows from the corresponding result for matrices, gives a similar result for a holomorphic family of elliptic operators.

[iml.24]

PROPOSITION 3.16. *Suppose $\Omega \subset \mathbb{C}$ is open and connected, $h(z)$ is holomorphic on Ω and $A(z) \in \Psi^{h(z)}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic and elliptic family. Then if $A(z)$ is invertible at one point it is invertible for all but a discrete set and its inverse is locally of the form $B(z) + H(z)$ where $B(z) \in \Psi^{-h(z)}_{\mathrm{iso}}(\mathbb{R}^n)$ is a locally holomorphic family and $H(z)$ is meromorphic with values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and has all residues of finite rank.*

PROOF. By the results of Section 3.13 we can choose, at least locally near any point, a holomorphic parametrix $B(z)$ for the family in $\Psi_{\mathrm{iso}}^{-h(z)}(\mathbb{R}^n)$. Suppose $p \in \Omega$ is a point at which $A(z)$ is invertible. Then the parametrix differs from the inverse by a smoothing operator, so modifying $B(z)$ by a constant smoothing operator if follows that $B(z)A(z) = \mathrm{Id} + E(z)$ is a holomorphic family in $\mathrm{Id} + \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ with $E(p) = 0$. It follows from the Fredholm alternative that $\mathrm{Id} + E(z)$ is invertible with holomorphic inverse near p. Thus, $A(z)$ is invertible for z in an open set around p; the set at which it is invertible is therefore open. Let $Z \subset \Omega$ be the closed set at which $A(z)$ is not invertible. If p is a boundary point of Z then it follows, using the notation above for a parametrix of $A(z)$ near p, that $\mathrm{Id} + E(z)$ is invertible near p and hence p is an isolated point of non-invertibility. Thus all boundary points of Z are isolated so, given that Ω is connected, it is discrete. The local structure of the inverse follows from this discussion. □

3.17. Resolvent and spectrum

For elliptic operators with index zero and a point of invertibility.

3.18. Residue trace[S.Residue.trace]

We have shown, in Proposition 3.14, the existence of a unique trace functional on the residual algebra $\Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$. We now follow ideas originating with Seeley, [**11**], and developed by Guillemin [**5**], [**6**] and Wodzicki [**14**], [**13**] to investigate the traces on the full algebra $\Psi_{\mathrm{iso}}^{\mathbb{Z}}(\mathbb{R}^n)$ of polyhomogeneous operators of integral order. We will prove the existence of a trace but defer until later the proof of its uniqueness.

Observe that for $A \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ the kernel can be written

$$A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\xi} a_L(x,\xi)\, d\xi$$

and hence the trace, from (3.85), becomes

$$\mathrm{Tr}(A) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} a_L(x,\xi)\, dx\, d\xi\,, \tag{3.103}$$ [Feb.17.2000.eq:1]

just the integral of the left-reduced symbol. In fact this is true for *any* amplitude (of order $-\infty$) representing A :

(3.104)

$$A = (2\pi)^{-n} \int e^{i(x-y)} a(x,y,\xi)\, d\xi \Longrightarrow \mathrm{Tr}(A) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} a(x,x,\xi)\, dx\, d\xi\,.$$ [Feb.17.2000.eq:2]

The integral in (3.103) extends by continuity to $a_L \in S^m_\infty(\mathbb{R}^{2n})$ provided $m < -2n$. Thus, as a functional,

$$\mathrm{Tr} : \Psi_{\infty,\mathrm{iso}}^{-2n-\epsilon}(\mathbb{R}^n) \to \mathbb{C},\ \text{ for any } \epsilon > 0. \tag{3.105}$$ [Feb.17.2000.eq:3]

To extend it further we need somehow to *regularize* the resultant divergent integral in (3.103) (and to pay the price in terms of properties). One elegant way to do this is to use a holomorphic family as discussed in Section 3.13. Notice that we are passing from the algebra-with-bounds in (3.105) to polyhomogeneous operators.

LEMMA 3.13. *If $A(z) \in \Psi_{\mathrm{iso}}^{z}(\mathbb{R}^n)$ is a holomorphic family then $f(z) = \mathrm{Tr}(A(z))$, defined by* (3.104) *when $\Re(z) < -2n$, extends to a meromorphic function of z with at most simple poles on the divisor*

$$\{-2n, -2n+1, \dots, -1, 0, 1, \dots\} \subset \mathbb{C}.$$

PROOF. We know that $A(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family if and only if its left-reduced symbol is of the form

$$\sigma_L(A(z)) = (1 + |x|^2 + |z|^2)^{z/2} a(z; x, \xi)$$

where $a(z; x, y)$ is an entire function with values in $S^0_{\mathrm{phg}}(\mathbb{R}^n)$. For $\Re z < -2n$ the trace of $A(z)$ is

$$f(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} (1 + |x|^2 + |\xi|^2)^{z/2} a(z; x, \xi)\, dx\, d\xi\,.$$

Consider the part of this integral on the ball

$$f_1(z) = (2\pi)^{-n} \int_{|x|^2+|\xi|^2 \le 1} (1 + |x|^2 + |\xi|^2)^{z/2} a(z, x, y)\, dx\, d\xi\,.$$

This is clearly an entire function of z, since the integrand is entire and the domain compact.

To analyze the remaining part $f_2(z) = f(z) - f_1(z)$ let us introduce polar coordinates

$$r = (|x|^2 + |\xi|^2)^{1/2},\ \theta = \frac{(x, \xi)}{r} \in \mathbb{S}^{2n-1}\,.$$

The integral, convergent in $\Re z < -2n$, becomes

$$f_2(z) = (2\pi)^{-n} \int_1^\infty \int_{\mathbb{S}^{2n-1}} (1 + r^2)^{z/2} \tilde{a}(z; r, \theta)\, d\theta r^{2n-1}\, dr.$$

Let us now pass to the radical compactification of $\mathbb{R}^{2n}$ or more prosaically, introduce $t = 1/r \in [0, 1]$ as variable of integration, so

$$f_2(z) = (2\pi)^{-n} \int_0^1 \int_{\mathbb{S}^{2n-1}} t^{-z} (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta)\, d\theta t^{-2n} \frac{dt}{t}\,.$$

Now the definition of $S^0_{\mathrm{phg}}(\mathbb{R}^{2n})$ reduces to the statement that

$$b(z; t, \theta) = (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta) \in \mathcal{C}^\infty(\mathbb{C} \times [0, 1] \times \mathbb{S}^{2n-1})^{[Feb.17.2000.eq:4]} \tag{3.106}$$

is holomorphic in z.

If we replace b by its Taylor series at $t = 0$ to high order,

$$b(z; t, \theta) = \sum_{j=0}^{k} \frac{t^j}{j!} b_j(z; \theta) + t^{k+1} b_{(k)}(z; t, \theta)\,,^{[Feb.17.2000.eq:5]} \tag{3.107}$$

where $b_{(k)}(z; t, \theta)$ has the same regularity (3.106), then $f_2(z)$ is decomposed as

$$f_2(z) = (2\pi)^{-n} \sum_{j=0}^{k} \int_0^1 \int_{\mathbb{S}^{2n-1}} \frac{t^{-z+j}}{j!} b_j(z; \theta) t^{-2n} \frac{dt}{t} + f_2^{(k)}(z).^{[Feb.17.2000.eq:6]} \tag{3.108}$$

The presence of this factor t^k in the remainder in (3.107) shows that $f_2^{(k)}(z)$ is holomorphic in $\Re z < -2n + k$. On the other hand the individual terms in the sum in (3.108) can be computed (for $\Re z < -2n$) as

$$(2\pi)^{-n} \left[\frac{t^{-z+j-2n}}{(-z + j - 2n)} \right]_0^1 \int_{\mathbb{S}^{2n-1}} b_j(z, \theta) \frac{d\theta}{j!}$$
$$= (2\pi)^{-n} \frac{1}{(z - j + 2n)} \int_{\mathbb{S}^{2n-1}} b_j(z, \theta) \frac{d\theta}{j!}.$$

Each of these terms extends to be meromorphic in the entire complex plane, with a simple pole (at most) at $z = -2n + j$. This shows that $f(z)$ has a meromorphic continuation as claimed. □

By this argument we have actually computed the residues of the analytic continuation of $\mathrm{Tr}(A(z))$ as

$$\lim_{z\to -2n+j} (z - j + 2n)\,\mathrm{Tr}(A(z)) = (2\pi)^{-n} \int_{\mathbb{S}^{2n-1}} a_j(\theta)\, d\theta \tag{3.109}$$ [1.2.2000.283]

when $a_j(\theta) \in \mathcal{C}^\infty(S^{2n-1})$ is the function occurring in the asymptotic expansion of the left symbol of $A(z)$:

$$\begin{gathered}\sigma_L(A(z)) \sim \sum_{j=0}^{\infty} (|x|^2 + |\xi|^2)^{z/2-j} \tilde{a}_j(z,\theta) \\ |x|^2 + |\xi|^2 \to \infty,\ \theta = \frac{(x,\xi)}{(|x|^2+|\xi|^2)^{1/2}},\ a_j(\theta) = \tilde{a}_j(-2n+j,\theta).\end{gathered} \tag{3.110}$$ [Feb.17.2000.eq:7]

More generally, if $m \in \mathbb{Z}$ and $A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family then

$$\begin{gathered}\mathrm{Tr}(A(z)) \text{ is meromorphic with at most} \\ \text{simple poles at } -2n - m + \mathbb{N}_0\,.\end{gathered}$$ [Feb.24.2000.eq:1]

Indeed this just follows by considering the family $A(z - m)$.

We are especially interested in the behavior at $z = 0$. Since the residue there is an integral of the term of order $-2n$, we know that

$$\begin{gathered}A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ holomorphic with } A(0) = 0 \\ \Longrightarrow \mathrm{Tr}(A(z)) \text{ is regular at } z = 0\,.\end{gathered} \tag{3.111}$$ [Feb.24.2000.eq:2]

This allows us to make the following definition:

$$\begin{gathered}\mathrm{Tr}_{\mathrm{Res}}(A) = \lim_{z\to 0} z\,\mathrm{Tr}(A(z)) \text{ if} \\ A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ is holomorphic with } A(0) = A\,.\end{gathered}$$ [Feb.24.2000.eq:3]

We know that such a holomorphic family exists, since we showed in Section 3.13 the existence of a holomorphic family $F(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ with $F(0) = \mathrm{Id}$; $A(z) = AF(z)$ is therefore an example. Similarly we know that $\mathrm{Tr}_{\mathrm{Res}}(A)$ is independent of the choice of holomorphic family $A(z)$ because of (3.111) applied to the difference, which vanishes at zero.

LEMMA 3.14. *The residue functional* $\mathrm{Tr}_{\mathrm{Res}}(A)$, $A \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$, *is a trace:*

$$\mathrm{Tr}_{\mathrm{Res}}([A,B]) = 0\ \forall\ A, B \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.112}$$ [Feb.24.2000.eq:4]

which vanishes on $\Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n)$ *and is given explicitly by*

$$\mathrm{Tr}_{\mathrm{Res}}(A) = (2\pi)^{-n} \int_{S^{2n-1}} a_{-2n}(\theta)\, d\theta \tag{3.113}$$ [Feb.24.2000.eq:5]

where $a_{-2n}(\theta)$ *is the term of order* $-2n$ *in the expansion of the left (or right) symbol of* a.

PROOF. We have already shown that $\mathrm{Tr}_{\mathrm{Res}}(A)$ is well-defined and (3.113) follows from (3.109) with $a_{-2n}(\theta)$ the term of order $-2n$ in the left-reduced symbol of $A = A(0)$. On the other hand, the same argument applies for the right-reduced symbol.

To see (3.112) just note that if $A(z)$ and $B(z)$ are holomorphic families with $A(0) = A$, and $B(0) = B$ then $C(z) = [A(z), B(z)]$ is a holomorphic family with $C(0) = [A, B]$. On the other hand, $\operatorname{Tr}(C(z)) = 0$ when $\Re z \gg 0$, so the analytic continuation of $\operatorname{Tr}(C(z))$ vanishes identically and (3.112) follows. $\square$

As we shall see below, $\operatorname{Tr}_{\mathrm{Res}}$ is the unique trace (up to a multiple of course) on $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$.

3.19. Exterior derivation

Let $A(z) \in \Psi^{z}_{\mathrm{iso}}(\mathbb{R}^n)$ be a holomorphic family with $A(0) = \mathrm{Id}$. Then

$$G(z) = A(z) \cdot A(-z) \in \Psi^{0}_{\mathrm{iso}}(\mathbb{R}^n)$$

is a holomorphic family of fixed order with $G(0) = \mathrm{Id}$. By analytic Fredholm theory

$$G^{-1}(z) \in \Psi^{0}_{\mathrm{iso}}(\mathbb{R}^n) \text{ is a meromorphic family with finite rank poles.} \tag{3.114}$$ [1.2.2000.284]

It follows that $A^{-1}(z) = A(-z)G^{-1}(z)$ is a meromorphic family of order $-z$ with at most finite rank poles and regular near 0. Set

$$\Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n) \ni B \mapsto A(z)BA^{-1}(z) = B(z)\,. \tag{3.115}$$ [1.2.2000.286]

Thus $B(z)$ is a meromorphic family of order m with $B(0) = B$. The derivative gives a linear map.

$$\Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n) \ni B \mapsto D_A B = \frac{d}{dz} A(z)BA^{-1}(z)|_{z=0} \in \Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n)\,. \tag{3.116}$$ [1.2.2000.285]

Proposition 3.17. *For any holomorphic family of order z, with $A(0) = \mathrm{Id}$, the map* (3.116)*, defined through* (3.115)*, is a derivation and for two choices of $A(z)$ the derivations differ by an inner derivation.*

Proof. Since

$$A(z)B_1B_2A^{-1}(z) = A(z)B_1A^{-1}(z)A(z)B_2A^{-1}(z)$$

it follows that

$$\frac{d}{dz} A(z)B_1B_2A^{-1}(z)|_{z=0} = (D_A B_1) \circ B_2 + B_1 \circ (D_A B_2)\,.$$ [1.2.2000.287]

If $A_1(z)$ and $A_2(z)$ are two holomorphic families of order z with $A_1(0) = A_2(0) = \mathrm{Id}$ then

$$A_2(z) = A_1(z)G(z)$$

when $G(z) \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a meromorphic family, with finite rank poles. Thus

$$\begin{aligned} A_2(z)BA_2^{-1}(z) &= A_1(z)G(z)BG^{-1}(z)A_1^{-1}(z) \\ &= A_1(z)BA^{-1}(z) + zA_1(z)H(z)A_1^{-1}(z)\,. \end{aligned}$$

Here $H(z) = (G(z)BG^{-1}(z) - B)/z$ is a holomorphic family of degree m with $H(0) = G'(0)B - BG'(0)$. Thus

$$\frac{d}{dz} A_2(z)BA_2^{-1}(z)|_{z=0} = \frac{d}{dz} A_1(z)BA^{-1}(z)|_{z=0} + [G'(0), B]$$

which shows that the two derivations differ by an inner derivation, which is to say commutation with an element of $\Psi^{0}_{\mathrm{iso}}(\mathbb{R}^n)$. $\square$

Note that in fact

$$D_A : \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \to \Psi^{m-1}_{\mathrm{iso}}(\mathbb{R}^n)\ \forall\, m$$

since the symbol of $A(z)BA^{-1}(z)$ is equal to the principal symbol of B for all z.

For the specific choice of $A(z) = H(z)$ given by

$$\sigma_L(H(z)) = (1 + |x|^2 + |\xi|^2)^{z/2}$$

we shall set

$$D_A B = D_H B\,.$$

Observe that $\frac{1}{2}\log(1 + |x|^2 + |\xi|^2) \in S^{\epsilon}_{\infty}(\mathbb{R}^{2n})\ \forall\ \epsilon > 0$. Thus $\log(1 + |x|^2 + |\xi|^2)$, defined by Weyl quantization, is an element of $\Psi^{-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for all $\epsilon > 0$. By differentiation the symbols satisfy

$$D_H B = [\frac{1}{2}\log(1 + |x|^2 + |D|^2), B] + [G, B]$$

where $G \in \Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus D_H is *not* itself an interior derivation. It is therefore an *exterior* derivation.

3.20. Regularized trace[S.Regularized.trace]

In Section 3.18 we defined the residue trace of B as the residue at $z = 0$ of the analytic continuation of $\mathrm{Tr}(BA(z))$, where $A(z)$ is a holomorphic family of order z with $A(0) = \mathrm{Id}$. Next we consider the functional

$$\overline{\mathrm{Tr}}_{\mathrm{A}}(B) = \lim_{z=0}(\mathrm{Tr}(BA(z)) - \frac{1}{z}\,\mathrm{Tr}_{\mathrm{Res}}(B))\,. \tag{3.117}$$ [Feb.24.2000.eq:E]

In contrast to the residue trace, $\overline{\mathrm{Tr}}_{\mathrm{A}}(z)$ *does* depend on the choice of analytic family $A(z)$.

Lemma 3.15. *If $A_i(z)$, $i = 1, 2$, are two holomorphic families of order z with $A_i(0) = \mathrm{Id}$ and $G'(0) = \frac{d}{dz}A_2(z)A_1^{-2}(z)|_{z=0}$ then*

$$\overline{\mathrm{Tr}}_{\mathrm{A}_2}(B) - \overline{\mathrm{Tr}}_{\mathrm{A}_1}(B) = \mathrm{Tr}_{\mathrm{Res}}(BG'(0))\,. \tag{3.118}$$ [Feb.24.2000.eq:F]

Proof. Writing $G(z) = A_2(z)A_1^{-1}(z)$, which is a meromorphic family of order 0 with $G(0) = \mathrm{Id}$,

$$\begin{aligned}\mathrm{Tr}(BA_2(z)) &= \mathrm{Tr}(BG(z)A_1(z))\\ &= \mathrm{Tr}(BA_1(z)) + z\,\mathrm{Tr}(BG'(0)A_1(z)) + z^2\,\mathrm{Tr}(H(z)A_1(z))\end{aligned}$$

where $H(z) = \frac{B}{z^2}(G(z) - \mathrm{Id} - zG'(0))$ is then meromorphic with only finite rank poles and is regular near $z = 0$. Thus the analytic continuation of $z^2\,\mathrm{Tr}(H(z)A(z))$ vanishes at zero from which (3.118) follows. □

This regularized trace $\overline{\mathrm{Tr}}_{\mathrm{A}}(B)$ therefore only depends on the first order, in z, term in $A(z)$ at $z = 0$. It is important to note that it is *not* itself a trace.

Lemma 3.16. *If $B_1,\ B_2 \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ then*

$$\overline{\mathrm{Tr}}_{\mathrm{A}}([B_1, B_2]) = \mathrm{Tr}_{\mathrm{Res}}(B_2 D_A B_1)\,. \tag{3.119}$$ [Feb.24.2000.eq:G]

Proof. Since $\overline{\mathrm{Tr}}_{\mathrm{A}}([B_1, B_2])$ is the regularized value at 0 of the analytic continuation of the trace of

$$\begin{aligned}(3.120)\quad B_1B_2A(z) - B_2B_1A(z) &= B_2[A(z), B_1] + [B_1, B_2A(z)]\\ &= B_2([A(z), B]A^{-1}(z))A_1(z) + [B_1B_2A(z)]\,.\end{aligned}$$ [Feb.24.2000.eq:H]

The second term on the right in (3.120) has zero trace before analytic continuation. Thus $\overline{\mathrm{Tr}}_A([B_1, B_2])$ is the regularized value of the analytic continuation of the trace of $Q(z)A(z)$ where

$$Q(z) = B_2[A(z), B_1]A^{-1}(z) = zD_AB_1 + z^2L(z)$$

with $L(z)$ meromorphic of fixed order and regular at $z = 0$. Thus (3.119) follows. □

Note that

$$(3.121)\qquad \mathrm{Tr}_{\mathrm{Res}}(D_AB) = 0\ \forall\ B \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$$ [Feb.24.2000.eq:H1]

and any family A. Indeed the residue trace is the residue of $z = 0$ of the analytic continuation of $\mathrm{Tr}(H(z)A(z))$ when $A(z)$ is any meromorphic family of fixed order with $H(0) = D_AB$. In particular we can take

$$H(z) = \frac{1}{z}(A(z)BA^{-1}(z) - B)\,.$$

Then $H(z)A(z) = \frac{1}{z}[A(z), B]$ so the trace vanishes before analytic continuation.

3.21. Projections

3.22. Complex powers

3.23. Index and invertibility[S.Index.and.invertibility]

We have already seen that the elliptic elements

$$(3.122)\qquad E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \subset \Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \hookrightarrow \mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N))$$ [1.2.2000.369]

define Fredholm operators. The *index* of such an operator

$$(3.123)\qquad \mathrm{Ind}(A) = \dim \mathrm{Nul}(A) - \dim \mathrm{Nul}(A^*)$$ [1.2.2000.370]

is a measure of its non-invertibility. Set

$$(3.124)\qquad E^0_{\mathrm{iso},k}(\mathbb{R}^n; \mathbb{C}^N) = \left\{A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N); \mathrm{Ind}(A) = k\right\},\ k \in \mathbb{Z}.$$ [1.2.2000.371]

[1.2.2000.372]

Proposition 3.18. *If $A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ and $\mathrm{Ind}(A) = 0$ then there exists $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ such that $A + E$ is invertible in $\mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N))$ and the inverse then lies in $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$.*

Proof. Let B be the generalized inverse of A, assumed to be elliptic. The assumption that $\mathrm{Ind}(A) = 0$ means that $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$ have the same dimension. Let $e_1, \cdots, e_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ and $f_1, \cdots, f_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ be bases of $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$. Then consider

$$(3.125)\qquad E = \sum_{j=1}^{p} f_j(x)\overline{e_j(y)} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N).$$

By construction E is an isomorphism (in fact an arbitrary one) between $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$. Thus $A+E$ is continuous, injective and surjective, hence has an inverse in

$\mathcal{B}(L^2(\mathbb{R}^n;\mathbb{C}^N))$. Indeed this inverse is $B+E^{-1}$ where E^{-1} is the inverse of E as a map from $\mathrm{Nul}(A)$ to $\mathrm{Nul}(A^*)$. This shows that A can be perturbed by a smoothing operator to be invertible. □

Let

$$G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\subset E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)\subset E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\subset \Psi 0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\tag{3.126}$$ [1.2.2000.374]

denote the group of the invertible elements (invertibility being either in $\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N)$ or in $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$) in the ring of elliptic elements of index 0.

[1.2.2000.373]

Corollary 3.7. *The first inclusion in* (3.126) *is dense in the topology of* $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

Proof. This follows from the proof of Proposition 3.18, since $A+sE$ is invertible for all $s\neq 0$. □

We next derive some simple formulæ for the index of an element of $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. First observe that the trace of a finite dimensional projection is its rank, the dimension of its range. Thus

$$\mathrm{Ind}(A)=\mathrm{Tr}(\Pi_{\mathrm{Nul}(A)})-\mathrm{Tr}(\Pi_{\mathrm{Nul}(A^*)})\tag{3.127}$$ [1.2.2000.376]

where the trace may be reinterpreted as the trace on smoothing operators. The identities, (3.34), satisfied by the generalized inverse of A shows that this can be rewritten

$$\mathrm{Ind}(A)=-\mathrm{Tr}(BA-\mathrm{Id})+\mathrm{Tr}(AB-\mathrm{Id})=\mathrm{Tr}([A,B]).\tag{3.128}$$ [1.2.2000.377]

Here $[A,B]=\Pi_{\mathrm{Nul}(A)}-\Pi_{\mathrm{Nul}(A^*)}$ is a smoothing operator, even though both A and B are elliptic of order 0.

[1.2.2000.378]

Lemma 3.17. *If $A\in E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ the identity* (3.128)*, which may be rewritten*

$$\mathrm{Ind}(A)=\mathrm{Tr}([A,B]),\tag{3.129}$$ [1.2.2000.379]

holds for any parametrix B.

Proof. If B' is a parametrix and B is the generalized inverse then $B'-B=E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. Thus

$$[A,B']=[A,B]+[A,E].$$

Since $\mathrm{Tr}([A,E]=0$, one of the arguments being a smoothing operator, (3.129) follows in general from the particular case (3.128). □

Note that it follows from (3.129) that $\mathrm{Ind}(A)=\mathrm{Ind}(A+E)$ if E is smoothing. In fact the index is even more stable than this as we shall see, since it is locally constant on $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. In any case this shows that

$$\begin{gathered}\mathrm{Ind}:\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\longrightarrow\mathbb{Z},\ \mathrm{Ind}(a)=\mathrm{Ind}(A)\text{ if }a=[A],\\ \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)=E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\\ \subset\mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)=\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\end{gathered}\tag{3.130}$$ [1.2.2000.381]

is well-defined.

The argument of the trace functional in (3.129) is a smoothing operator, but we may still rewrite the formula in terms of the regularized trace, with respect to the standard regularizer $H(z)$ with left symbol $(1+|x|^2+|\xi|^2)^{\frac{z}{2}}$. The advantage of doing so is that we can then use the trace defect formula (3.119). Thus for any elliptic isotropic operator of order 0

$$\operatorname{Ind}(A) = \operatorname{Tr}_{\mathrm{Res}}(BD_HA). \tag{3.131}$$ [1.2.2000.380]

Here B is a parametrix for A. The residue trace is actually a functional

$$\operatorname{Tr}_{\mathrm{Res}} : \mathcal{A}^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C},$$

so if we write a^{-1} for the inverse of a in the ring $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then

$$\operatorname{Ind}(a) = \operatorname{Tr}_{\mathrm{Res}}(a^{-1}D_Ha),\ D_H : \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \tag{3.132}$$ [1.2.2000.382]

being the induced derivation (since D_H clearly preserves the ideal $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

From this simple formula we can easily deduce two elementary properties of elliptic operators. These actually hold in general for Fredholm operators, although the proofs here are not valid in that generality. Namely

$$\operatorname{Ind} : \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z} \text{ is locally constant and} \tag{3.133}$$ [1.2.2000.383]

$$\operatorname{Ind}(a_1a_2) = \operatorname{Ind}(a_1) + \operatorname{Ind}(a_2)\ \forall\ a_1,\ a_2 \in \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N). \tag{3.134}$$ [1.2.2000.384]

The first of these follows the continuity of the formula (3.132) since under deformation of a in $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ the inverse a^{-1} varies continuously, so Ind is continuous and integer-valued, hence locally constant. Similarly the second, logarithmic additivity, property follows from the fact that D_H is a derivation, so

$$D_H(a_1a_2) = (D_Ha_1)a_2 + a_1D_Ha_2$$

and the the trace property of $\operatorname{Tr}_{\mathrm{Res}}$ which shows that

(3.135)
$$\begin{aligned}\operatorname{Ind}(a_1a_2) = \operatorname{Tr}_{\mathrm{Res}}((a_1a_2)^{-1}D_H(a_1a_2) &= \operatorname{Tr}(a_2^{-1}a_1^{-1}\,((D_Ha_1)a_2 + a_1D_Ha_2)\\ &= \operatorname{Tr}(a_2^{-1}a_1^{-1}(D_Ha_1)a_2) + \operatorname{Tr}(a_2^{-1}D_Ha_2) = \operatorname{Ind}(a_1) + \operatorname{Ind}(a_2).\end{aligned}$$ [1.2.2000.385]

3.24. Variation 1-form

In the previous section we have seen that the index

$$\operatorname{Ind} : E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z} \tag{3.136}$$ [1.2.2000.386]

is a multiplicative map which is the obstruction to perturbative invertibility. In the next two sections we will derive a closely related obstruction to the perturbative invertibility of a family of elliptic operators. Thus, suppose

$$Y \ni y \longmapsto A_y \in E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N) \tag{3.137}$$ [1.2.2000.387]

is a family of elliptic operators depending smoothly on a parameter in the compact manifold Y. We are interested in the *families perturbative invertibility question.* That is, does there exist a smooth family

$$Y \ni y \longmapsto E_y \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \text{ such that } (A_y + E_y) \in G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\ \forall\ y. \tag{3.138}$$ [1.2.2000.388]

We have assumed that the operators have index zero since this is necessary (and sufficient) for E_y to exist for any one $y \in Y$. Thus the issue is the smoothness (really just the continuity) of the perturbation E_y.

We shall start by essentially writing down such a putative obstruction directly and then subsequently we shall investigate its topological origins.

[1.2.2000.389]

PROPOSITION 3.19. *If a smooth family* (3.137), *parameterized by a compact manifold* Y, *is perturbatively invariant in the sense that there is a smooth family as in* (3.138), *then the closed* 2*-form on* Y

$$\beta = \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \cdot a_y^{-1}D_H a_y) \in \mathcal{C}^\infty(Y;\Lambda^2),\qquad a_y = [A_y] \in \mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N), \tag{3.139}$$
[1.2.2000.390]

is exact.

PROOF. Note first that β is indeed a smooth form, since the full symbolic inverse depends smoothly on parameters. Next we show that β is always closed. The 1-forms $a_y^{-1}d_y a_y a_y^{-1}$ and da_y are exact so differentiating directly gives

$$\begin{aligned} d\beta &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge d(a_y^{-1}D_H a_y)) \\ &= -\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y^{-1}D_H a_y)) \\ &\quad +\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}D_H(da_y)) \\ &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge D_H(a_y^{-1}da_y)). \end{aligned} \tag{3.140}$$
[1.2.2000.391]

Using the trace property and the commutativity of a 2-form with other forms the last expression can be written

$$\frac{1}{6}\operatorname{Tr}_{\mathrm{Res}}(D_H(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}da_y)) = 0 \tag{3.141}$$
[1.2.2000.392]

by property (3.121) of the residue trace.

Now, suppose that a smooth perturbation as in (3.138) does exist. We can replace A_y by $A_y + E_y$ without affecting β, since the residue trace vanishes on the ideal of smoothing operators. Thus we can assume that A_y itself is invertible. Then consider the 1-form defined using the regularized trace

$$\overline{\alpha} = \overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}d_y A_y). \tag{3.142}$$
[1.2.2000.393]

This is an extension of the 1-form $d\log\det_F$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. The extension is not in general closed, because the regularized trace does not satisfy the trace condition. Using the stanadard formula for the variation of the inverse, $dA_y^{-1} = -A_y^{-1}dA_yA_y^{-1}$, the exterior derivative is the 2-form

$$d\overline{\alpha} = -\overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}(d_yA)A_y^{-1}d_yA_y). \tag{3.143}$$
[1.2.2000.394]

The 2-form argument is a commutator. Indeed, in terms of local coordinates we can write

$$\begin{aligned} A_y^{-1}(d_yA)A_y^{-1}d_yA_y &= \sum_{j,k=1}^{p} A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k})dy_j\wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}\left(A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k}) - A_y^{-1}(\frac{\partial A}{\partial y_k})A_y^{-1}(\frac{\partial A}{\partial y_j})\right)dy_j\wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}[A_y^{-1}(\frac{\partial A}{\partial y_j}), A_y^{-1}(\frac{\partial A}{\partial y_k})]dy_j\wedge dy_k \end{aligned}$$

Applying the trace defect formula (3.119) shows that

$$d\overline{\alpha} = -\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge D_H(A_y^{-1}d_yA_y)\right), \tag{3.144}$$ [1.2.2000.395]

locally and hence globally.

Expanding the action of the derivation D_H gives

$$d\overline{\alpha} = \beta - \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge A_y^{-1}d_y(D_HA_y)\right) = \beta - d\gamma, \text{ where}$$
$$\gamma = \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge .A_y^{-1}D_HA_y\right). \tag{3.145}$$ [1.2.2000.396]

We conclude that if A_y has an invertible lift then β is exact. □

Note that the form γ in (3.145) is well-defined as a form on $\mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)$, and is independent of the perturbation. Thus the cohomology class which we have constructed as the obstruction to perturbative invertibility can be written

$$[\beta] = [\beta - d\gamma] \in H^2(\mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)). \tag{3.146}$$ [1.2.2000.397]

3.25. Determinant bundle

To better explain the topological origin of the cohomology class (3.146) we construct the determinant bundle. This was originally introduced for families of Dirac operators by Quillen [**10**]. Recall that the Fredholm determinant is a character

$$\det_{\mathrm{Fr}} : \mathrm{Id} + \Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C},$$
$$\det_{\mathrm{Fr}}(AB) = \det_{\mathrm{Fr}}(A)\det_{\mathrm{Fr}}(B) \forall\ A,\ B \in \mathrm{Id} + \Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N). \tag{3.147}$$ [1.2.2000.398]

As we shall see, it is not possible to extend the Fredholm determinant as a multiplicative function to $G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$, essentially because of the non-extendibility of the trace.

However in trying to extend the determinant we can consider the possible values it would take on a point $A \in G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ as the set of pairs (A,z), $z \in \mathbb{C}$. Thus we simple consider the product

$$D^0 = G^0 \times \mathbb{C}, \tag{3.148}$$ [1.2.2000.399]

where from now on we simplify the notation and write $G^0 = G_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ etc. Although it is not reasonable to expect full multiplicative of the determinant, it is more reasonable to expect the determinant of $A(\mathrm{Id}+B)$, $B \in \Psi^{-2n-1}$ to be related to the product of determinants. Thus it is natural to identify pairs in D^0,

$$(A,z) \sim_p (A',z') \text{ if}$$
$$A, A' \in G^0,\ A' = A(\mathrm{Id}+B),\ z' = \det_{\mathrm{Fr}}(\mathrm{Id}+B)z,\ B \in \Psi^p,\ p < -2n. \tag{3.149}$$ [1.2.2000.400]

The equivalence relations here are slightly different, depending on p. In all cases the action of the determinant is linear, so the quotient is a line bundle.

[1.2.2000.401]

Lemma 3.18. *For any integer $p < -2n$, and also $p = -\infty$, the quotient*

$$\mathcal{D}^0_p = D^0/\sim_p \tag{3.150}$$ [1.2.2000.402]

is a smooth line bundle over $\mathcal{G}^0_p = G^0/G^p$.

PROOF. The projection is just the quotient in the first factor and this clearly defines a commutative square

$$
\begin{tikzcd}
D^0 \arrow[r, "{[\sim_p]}"] \arrow[d, "\pi"] & \mathcal{D}^0_p \arrow[d, "\pi"] \\
G^0 \arrow[r, "/G^p"] & \mathcal{G}^0_p .
\end{tikzcd}
\tag{3.151}
$$
[1.2.2000.403]

□

3.26. Index bundle

3.27. Index formulæ

susceptible

3.28. Isotropic essential support

3.29. Isotropic wavefront set

3.30. Isotropic FBI transform

3.31. Problems[S.Problems.3]

[21.2.1998.121]

PROBLEM 3.1. Define the *isotropic* Sobolev spaces of integral order by

$$
H^k_{\mathrm{iso}}(\mathbb{R}^n) = \begin{cases} \left\{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| + |\beta| \le k\right\} & k \in \mathbb{N} \\ \left\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum\limits_{|\alpha|+|\beta|\le -k} x^\alpha D^\beta_x u_{\alpha,\beta},\ u_{\alpha,\beta} \in L^2(\mathbb{R}^n)\right\} & k \in -\mathbb{N}. \end{cases}
\tag{3.152}
$$
[21.2.1998.122]

Show that if $A \in \Psi^p_{\mathrm{iso}}(\mathbb{R}^n)$ with p an integer, then $A : H^k_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{k-p}_{\mathrm{iso}}(\mathbb{R}^n)$ for any integral k. Deduce (using the properties of elliptic isotropic operators) that the general definition

$$
H^m_{\mathrm{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n),\ \forall\ A \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)\right\},\ m \in \mathbb{R}
\tag{3.153}
$$
[21.2.1998.123]

is consistent with (3.152) and has the properties

$$
A \in \Psi^M_{\mathrm{iso}}(\mathbb{R}^n) \Longrightarrow A : H^m_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{m-M}_{\mathrm{iso}}(\mathbb{R}^n),
\tag{3.154}
$$
[21.2.1998.124]

$$
\bigcap_m H^m_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n),\ \bigcup_m H^m_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)
\tag{3.155}
$$
[21.2.1998.126]

$$
A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n),\ u \in \mathcal{S}'(\mathbb{R}^n),\ Au \in H^{m'}(\mathbb{R}^n) \Longrightarrow u \in H^{m'-m}(\mathbb{R}^n),
\tag{3.156}
$$
[21.2.1998.125]

[21.2.1998.127]

PROBLEM 3.2. Show that if $\epsilon > 0$ then

$$
H^\epsilon_{\mathrm{iso}}(\mathbb{R}^n) \subsetneq (1 + |x|)^{-\epsilon} L^2(\mathbb{R}^n) \cap H^\epsilon(\mathbb{R}^n)
$$

Deduce that $H^\epsilon_{\mathrm{iso}}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is a compact inclusion (i.e. the image of a bounded set is precompact).

[21.2.1998.128]

PROBLEM 3.3. Using Problem 3.2, or otherwise, show that each element of $\Psi^{-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$, $\epsilon > 0$, defines a compact operator on $L^2(\mathbb{R}^n)$.

[21.2.1998.129]

PROBLEM 3.4. Show that if $E \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ then there exists $F \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ such that

$$(\mathrm{Id} + E)(\mathrm{Id} + F) = \mathrm{Id}_G \text{ with } G \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n) \text{ of finite rank,}$$

that is, $G \cdot \mathcal{S}(\mathbb{R}^n)$ is finite dimensional.

[21.2.1998.130]

PROBLEM 3.5. Using Problem 3.4 show that an elliptic element $A \in \Psi_{\text{iso}}^{m}(\mathbb{R}^n)$ has a parametrix $B \in \Psi_{\text{iso}}^{-m}(\mathbb{R}^n)$ *up to finite rank error*; that is, such that $A \circ B - \mathrm{Id}$ and $B \circ A - \mathrm{Id}$ are finite rank elements of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$. Deduce that such an elliptic A defines a Fredholm operator

$$A : H_{\text{iso}}^{M}(\mathbb{R}^n) \longrightarrow H_{\text{iso}}^{M-m}(\mathbb{R}^n)$$

for any M. [The requirements for an operator A between Hilbert spaces to be Fredholm are that it be bounded, have finite-dimensional null space and closed range with a finite-dimensional complement.]

[21.2.1998.111]

PROBLEM 3.6. [The harmonic oscillator] Show that the 'harmonic oscillator'

$$H = |D|^2 + |x|^2, \ Hu = \sum_{j=1}^{n} D_j^2 u + |x|^2 u,$$

is an elliptic element of $\Psi_{\text{iso}}^{2}(\mathbb{R}^n)$. Consider the 'creation' and 'annihilation' operators

$$C_j = D_j + ix_j, \ A_j = D_j - ix_j = C_j^*, \tag{3.157}$$ [21.2.1998.131]

and show that

$$H = \sum_{j=1}^{n} C_j A_j + n = \sum_{j=1}^{n} A_j C_j - n, \tag{3.158}$$

$$[A_j, H] = 2A_j, \ [C_j, H] = -2C_j, \ [C_l, C_j] = 0, \ [A_l, A_j] = 0, \ [A_l, C_j] = 2\delta_{lk}\,\mathrm{Id},$$ [21.2.1998.132]

where $[A, B] = A \circ B - B \circ A$ is the commutator bracket and δ_{lk} is the Kronecker symbol. Knowing that $(H - \lambda)u = 0$, for $\lambda \in \mathbb{C}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{S}(\mathbb{R}^n)$ (why?) show that

$$E_\lambda = \{u \in \mathcal{S}'(\mathbb{R}^n); (H - \lambda)u = 0\} \neq \{0\} \Longleftrightarrow \lambda \in n + 2\mathbb{N}_0 \tag{3.159}$$ [21.2.1998.133]

$$\text{and } E_{-n+2k} = \left\{ \sum_{|\alpha|=k} c_\alpha C^\alpha \exp(-|x|^2/2), \ c_\alpha \in \mathbb{C} \right\}, \ k \in \mathbb{N}_0. \tag{3.160}$$ [21.2.1998.134]

[1.2.2000.406]

PROBLEM 3.7. [Definition of determinant of matrices.]

[$iml.6$]

PROBLEM 3.8. [Proof that $d\alpha = 0$ in (3.93).] To prove that the 1-form is closed it suffices to show that it is closed when restricted to any 2-dimensional submanifold. Thus we may suppose that $A = A(s,t)$ depends on 2 parameters. In terms of these parameters

$$\alpha = \mathrm{Tr}(A(s,t)^{-1}\frac{dA(s,t)}{ds})ds + \mathrm{Tr}(A(s,t)^{-1}\frac{dA(s,t)}{dt})dt. \tag{3.161}$$ [$iml.7$]

Show that the exterior derivative can be written

$$d\alpha = \mathrm{Tr}([A(s,t)^{-1}\frac{dA(s,t)}{dt}, A(s,t)^{-1}\frac{dA(s,t)}{ds}])ds \wedge dt \tag{3.162}$$ [iml.8]

and hence that it vanishes.

[21.2.1998.137]

PROBLEM 3.9. If E and F are vector spaces, show that the space of operators $\Psi^m_{\text{iso}}(\mathbb{R}^n; E, F)$ from $\mathcal{S}'(\mathbb{R}^n; E)$ to $\mathcal{S}'(\mathbb{R}^n; F)$ is well-defined as the matrices with entries in $\Psi^m_{\text{iso}}(\mathbb{R}^n)$ for any choice of bases of E and F.

[1.2.2000.349]

PROBLEM 3.10. Necessity of ellipticity for a psuedodifferential operator to be Fredholm on the isotropic Sobolev spaces.

1. Reduce to the case of operators of order 0.
2. Construct a sequence in L^2 such that $\|u_n\| = 1$, $u_n \to 0$ weakly and $Au_n \to 0$ strongly in L^2.

[21.2.1998.135]

PROBLEM 3.11. [Koszul complex] Consider the form bundles over $\mathbb{R}^n$. That is $\Lambda^k\mathbb{R}^n$ is the vector space of dimension $\binom{n}{k}$ consisting of the totally antisymmetric k-linear forms on $\mathbb{R}^n$. If $e_1, e_2, \dots, e_n$ is the standard basis for $\mathbb{R}^n$ then for a k-tuple α e^α defined on basis elements by

$$e^\alpha(e_{i_1}, \dots, e_{i_k}) = \prod_{j=1}^k \delta_{1_j \alpha_j}$$

extends uniquely to a k-linear map. Elements $dx^\alpha \in \Lambda^k\mathbb{R}^n$ are defined by the total antisymmetrization of the e^α. Explicitly,

$$dx^\alpha(v_1, \dots, v_k) = \sum_\pi \operatorname{sgn}\pi e^\alpha(v_{\pi_1}, \dots, v_{\pi_n})$$

where the sum is over permutations π of $\{1, \dots, n\}$ and $\operatorname{sgn}\pi$ is the parity of π. The dx^α for strictly increasing k-tuples α of elements of $\{1, \dots, n\}$ give a basis for $\Lambda^k\mathbb{R}^n$. The wedge product is defined by $dx^\alpha \wedge dx^\beta = dx^{\alpha,\beta}$.

Now let $\mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ be the tensor product, that is $u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ is a finite sum

$$u = \sum_\alpha u_\alpha dx^\alpha. \tag{3.163}$$ [21.2.1998.136]

The annihilation operators in (3.157) define an operator, for each k,

$$D : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k+1}),\ Du = \sum_{j=1}^n A_j u_\alpha dx^j \wedge dx^\alpha.$$

Show that $D^2 = 0$. Define inner products on the $\Lambda^k\mathbb{R}^n$ by declaring the basis introduced above to be orthonormal. Show that the adjoint of D, defined with respect to these inner products and the L^2 pairing is

$$D^* : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ D^*u = \sum_{j=1}^n C_j u_\alpha \iota_j dx^\alpha.$$

Here, ι_j is 'contraction with e_j;' it is the adjoint of $dx^j \wedge$. Show that $D + D^*$ is an elliptic element of $\Psi^1_{\text{iso}}(\mathbb{R}^n; \Lambda^*)$. Maybe using Problem 3.6 show that the null space of $D + D^*$ on $\mathcal{S}'(\mathbb{R}^n; \Lambda^*\mathbb{R}^n)$ is 1-dimensional. Deduce that

$$
\begin{aligned}
(3.164)\quad & \{u \in \mathcal{S}'(\mathbb{R}^n); Du = 0\} = \mathbb{C}\exp(-|x|^2/2), \\
& \{u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k); Du = 0\} = (\mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ k \geq 1.
\end{aligned}
$$ [21.2.1998.138]

Observe that, as an operator from $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\text{odd}})$ to $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\text{even}})$, $D + D^*$ is an elliptic element of $\Psi^1_{\text{iso}}(\mathbb{R}^n; \Lambda^{\text{odd}}, \Lambda^{\text{even}})$ and has index 1.

[22.2.1998.141]

PROBLEM 3.12. [Isotropic essential support] For an element of $S^m(\mathbb{R}^n)$ define (isotropic) essential support, or operator wavefront set, of $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$ by

$$\mathrm{WF}_{\text{iso}}(A) = \operatorname{cone\,supp}(\sigma_L(A)) \subset \mathbb{R}^{2n} \setminus \{0\}. \tag{3.165}$$ [22.2.1998.145]

Show that $\mathrm{WF}_{\text{iso}}(A) = \operatorname{cone\,supp}(\sigma_L(A))$ and check the following

$$\mathrm{WF}'_{\text{iso}}(A+B) \cup \mathrm{WF}'_{\text{iso}}(A \circ B) \subset \mathrm{WF}'_{\text{iso}}(A) \cap \mathrm{WF}'_{\text{iso}}(B), \tag{3.166}$$ [22.2.1998.149]

$$\mathrm{WF}'_{\text{iso}}(A) = \emptyset \Longleftrightarrow A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n). \tag{3.167}$$ [22.2.1998.146]

[22.2.1998.150]

PROBLEM 3.13. [Isotropic partition of unity] Show that if $U_i \subset \mathbb{S}^{n-1}$ is an open cover of the unit sphere and $\tilde{U}_i = \{Z \in \mathbb{R}^{2n} \setminus \{0\}; \frac{Z}{|Z|} \in U_i\}$ is the corresponding *conic* open cover of $\mathbb{R}^{2n} \setminus \{0\}$ then there exist (finitely many) operators $A_i \in \Psi^0_{\text{iso}}(\mathbb{R}^n)$ with $\mathrm{WF}'_{\text{iso}}(A_i) \subset \tilde{U}_i$, such that

$$\mathrm{Id} - \sum_i A_i \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n). \tag{3.168}$$ [22.2.1998.151]

[22.2.1998.152]

PROBLEM 3.14. Suppose $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$, is elliptic and has index zero as an operator on $\mathcal{S}'(\mathbb{R}^n)$. Show that there exists $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ such that $A + E$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$.

[22.2.1998.147]

PROBLEM 3.15. [Isotropic wave front set] For $u \in \mathcal{S}'(\mathbb{R}^n)$ define

$$\mathrm{WF}_{\text{iso}}(u) = \bigcap \left\{\mathrm{WF}'_{\text{iso}}(A); A \in \Psi^0_{\text{iso}}(\mathbb{R}^n),\ Au \in \mathcal{S}(\mathbb{R}^n)\right\}. \tag{3.169}$$ [22.2.1998.148]