CHAPTER 7

Localization and composition

Lecture 7: 6 October, 2005

L7.1. Localization. Finally I will connect the definition of pseudodifferential operators made here with the more standard approach, starting with a local definition on Euclidean space and proceeding by superposition. To break a pseudodifferential operator up into pieces it is convenient to use partitions of unity of the following type.

LEMMA 13. *If $\{U_i\}$ is an open cover of a compact manifold there is a partition of unity $\{\phi_{ij}\}$ subordinate to the cover, so*

$$0 \le \phi_{ij} \le 1, \ \forall\ i,j, \ \sum_{i,j} \phi_{ij} = 1, \ \operatorname{supp}(\phi_{ij}) \subset U_i,$$

which also satisfies

$$\operatorname{supp}(\phi_{ij}) \cap \operatorname{supp}(\phi_{i'j'}) \neq \emptyset \Longrightarrow \operatorname{supp}(\phi_{ij}) \cup \operatorname{supp}(\phi_{i'j'}) \subset U_i \cap U_{i'}. \tag{L7.1}$$

PROOF. First choose a partition of unity χ_i subordinate to the open cover $\{U_i\}$. Then each point $p \in X$ has an open neighbourhood V_p with the property

$$V_p \cap \operatorname{supp}(\chi_i) \neq \emptyset \Longrightarrow V_p \subset U_i. \tag{L7.2}$$

In fact we could take V_p to be the intersection of the U_i containing p. Pass from the V_p to a finite subcover, V_j, and choose a partition of unity ψ_j subordinate to this cover. Then set $\phi_{ij} = \chi_i\psi_j$. This is a partition of unity and the intersection condition in (L7.1) implies that the supports of ψ_j and $\chi_{i'}$ must meet, as well as those of $\psi_{j'}$ and χ_i. By (L7.2) this implies that $\operatorname{supp}(\psi_j) \subset U_{i'}$ and $\operatorname{supp}(\psi_{j'}) \subset U_i$ from which (L7.1) follows. □

We can use this to localize a pseudodifferential operator with respect to an open cover of X. Namely if $A \in \Psi^m(X; E, F)$ consider the decomposition obtained by mutliplying by the ϕ_{ij} on both the left and the right. That is, using the partition of unity $\phi_{ij}(x)\phi_{i'j'}(y)$ on X^2. This decomposes A (using the $\mathcal{C}^\infty$ module property) as a finite sum

$$A = \sum_{i,j,i',j'} \phi_{ij} A \phi_{i'j'} \tag{L7.3}$$

where we are thinking of the ϕ_{ij} as operators on $\mathcal{C}^\infty$ spaces, so (L7.3) is a composition of operators. The support of each term in (L7.3) is contained in $U_i \times U_{i'}$ but more importantly the support can only meet the diagonal if

(L7.4)
$$(\mathrm{supp}(\phi_{ij}) \times X) \cap (X \times \mathrm{supp}(\phi_{i'j'}) \cap \mathrm{Diag} \neq \emptyset \Longrightarrow \mathrm{supp}(\phi_{ij}) \cap \mathrm{supp}(\phi_{i'j'}) \neq \emptyset.$$

So, if we use the partition of unity from Lemma L7.2, then

$$\mathrm{supp}(\phi_{ij} A \phi_{i'j'}) \cap \mathrm{Diag} \neq \emptyset \Longrightarrow \mathrm{supp}(\phi_{ij} A \phi_{i'j'}) \subset U_i \times U_i. \tag{L7.5}$$

So, given an open cover $\{U_i\}$ of X we may decompose A into a sum of pseudodifferential operators of the same order

$$A = \sum_i A_i + A',\ \mathrm{supp}(A_i) \subset U_i \times U_i,\ A' \in \Psi^{-\infty}(X; E, F) \tag{L7.6}$$

where the last term comes from all the pieces which have support not meeting the diagonal.

L7.2. Local normal fibrations. In particular we can assume that the open cover $\{U_i\}$ with respect to which we get a decomposition (L7.6) consists of coordinate patches over each of which the bundles E and F are trivialized. Then the kernel of each A_i is a matrix of conormal distributions, with compact support and of order m, with respect to the diagonal in $U_i \times U_i$. The coordinate system identifies U_i with an open set U_i' in $\mathbb{R}^n$, $n = \dim X$. The density bundle on X is locally trivialized by the coordinate density $|dx|$ so it sufficies to consider 'scalar' pseudodifferential operators with kernels compactly supported on $\mathbb{R}^n \times \mathbb{R}^n$. This indeed is a typical starting point for the definition of pseudodifferential operators.

To specify the kernel as the inverse Fourier transform of a symbol we also need to choose a normal fibration of the diagonal

$$\mathrm{Diag}(\mathbb{R}^n) = \{x = y\} \subset \mathbb{R}^n_x \times \mathbb{R}^n_y. \tag{L7.7}$$

There are three standard choices for the normal fibration, which I will call the 'left' fibration, the 'right' fibration and the 'Weyl' fibration. These each give a global identification of the whole of $\mathbb{R}^{2n}$, as a neighbourhood of the diagonal, with $\mathbb{R}^n \times \mathbb{R}^n$, thought of as the normal bundle to the diagonal.

So first we have to identify the normal bundle to the diagonal. This is naturally the quotient of the tangent bundle to $\mathbb{R}^{2n}$, restricted to Diag, by the tangent bundle to Diag. The latter is easy to describe, namely

$$T\,\mathrm{Diag} = \{((x,x),(v,v)); (x,v) \in \mathbb{R}^{2n}\} \equiv \mathbb{R}^n \times \mathbb{R}^n \equiv \{(x,v) \in T\mathbb{R}^n\} \tag{L7.8}$$

where this identification is canonical. So the normal bundle can be identified with any subbundle of $T_{\mathrm{Diag}}\mathbb{R}^{2n}$ which is transversal to $T\,\mathrm{Diag}$. The standard choice is to take the 'left tangent bundle'

$$T\mathbb{R}^n \ni (x,w) \longmapsto ((x,x),(w,0)) \in T_{\mathrm{Diag}}\mathbb{R}^n \longrightarrow N\,\mathrm{Diag}. \tag{L7.9}$$

Notice that this is not really canonical. Namely we could 'just as well' take the right tangent vectors (but DO NOT DO THIS if you are easily confused)

$$T\mathbb{R}^n \ni (x,w) \longmapsto ((x,x),(0,w)) \in T_{\mathrm{Diag}}\mathbb{R}^{2n} \longrightarrow N\,\mathrm{Diag}.$$

The trouble is that modulo the tangent bundle to the diagonal $(0,w) - (w,w) = (-w,0)$ so this is almost the same identification but has the sign reversed. The identification (L7.9) is universally adopted, basically in the same sense that one writes compositions of operators on the left, i.e. AB means first apply B then A.

Once we have adopted (L7.9) as our identification of the normal bundle to the diagonal with the tangent bundle to the manifold (this works on a manifold as well) then there are still choices for the normal fibration. Now of course they correspond to maps from $\mathbb{R}^{2n}$ to $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with the right properties. The ones we consider each induce a linear isomorphism (linear in fact in all variables, not just the fibre variables). These are the left, the right and the Weyl fibrations:-

$$\begin{aligned} f_L &: \mathbb{R}^{2n} \ni (x,y) \longrightarrow (x, x-y) \in T\mathbb{R}^n \\ f_R &: \mathbb{R}^{2n} \ni (x,y) \longrightarrow (y, x-y) \in T\mathbb{R}^n \\ f_W &: \mathbb{R}^{2n} \ni (x,y) \longrightarrow (\frac{x+y}{2}, x-y) \in T\mathbb{R}^n. \end{aligned} \tag{L7.10}$$

Thus, for the left fibration we fix the variable x, so with the standard picture of x, y-space the fibres are the verticals, but we take the linear variable on each fibre which is $x - y$, the x being constant normalizes this to be zero at the point (x, x) on the diagonal, but the 'variable' is $-y$. This comes about because of the standard identification of the normal bundle to the diagonal with the tangent bundle. The right fibration is similar, except that y is held fixed, the fibres are 'horizontal' and the variable on them is still $x - y$. For the Weyl fibration, which I will not use for the moment, we hold $x + y$ fixed and the fibre variable is still $x - y$. There are plenty of other possibilities, but these are the usual ones.

So, what does our kernel $A \in \Psi^m(X)$, supported in a coordinate patch, look like with respect to these fibrations? It is always the inverse Fourier transform of a classical symbol, so the three representations (of the one kernel) are

$$\begin{aligned} A(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x,\xi) d\xi |dy|, \\ A(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_R(y,\xi) d\xi |dy|, \\ A(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_W(\frac{x+y}{2},\xi) d\xi |dy|. \end{aligned} \tag{L7.11}$$

Here $|dy|$ is the coordinate trivialization of the right density bundle. In all three cases the ampllitude lies in $\rho^{-m}\mathcal{C}^\infty_c(\mathbb{R}^n \times \overline{\mathbb{R}^n})$.

For the moment, we are most interested in the two 'extreme' representations, the left and right representations. As noted above, in each case we are holding one of the variables x or y fixed. This means that there is a close relationship between the Fourier transform and the operator.

Lemma 14. *The left representation of a pseudodifferential operator with compactly supported kernel on $\mathbb{R}^n$ puts the operator in the form*

$$Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a_L(x,\xi) \hat{f}(\xi) d\xi, \ \forall\ f \in \mathcal{S}(\mathbb{R}^n), \tag{L7.12}$$

and similarly the right representation gives

$$\widehat{Af}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} a_R(y,\xi) f(y) dy. \tag{L7.13}$$

Proof. □

L7.3. Composition. Almost as an immediate corollary of the representations (L7.12) and (L7.13) we deduce the basic composition property of pseudodifferential operators.

PROPOSITION 13. *If $A \in \Psi^m(X;E,F)$ and $B \in \Psi^{m'}(X;F,G)$ for complex vector bundles, E, F and G over a compact manifold X then as an operator*

$$BA : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;G),\ BA \in \Psi^{m+m'}(X;E,G). \tag{L7.14}$$

Furthermore

$$\sigma_{m+m'}(BA) = \sigma_{m'}(B)\sigma_m(A). \tag{L7.15}$$

PROOF. First we start with the 'easy case' where $m = -\infty$ or $m' = -\infty$ and one of the operators is smoothing. The composition is then very closely related to the action of pseudodifferential operators on smooth sections. In fact below I observe that it can be deduced directly from the continuity of this action after localizing.

However, one can also proceed directly and globally. I want to point out this argument, although I give a simpler alternative below, because it leads to an interesting geometric question which I will consider later.

Recall that we showed that $A \in \Psi^m(X;E,F)$ defines a map

$$A : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F)$$

by working on the kernel level. Namely we define the map (L7.3) by proceeding in steps. First lift an element $u \in \mathcal{C}^\infty(X;E)$ to the section $\pi_R^* u \in \mathcal{C}^\infty(X^2;\pi_R^* E)$ which is independent of the left, x, variable. Then use the $\mathcal{C}^\infty$-module property to 'multiply' the kernel by this smooth section (and compose in the bundle) to get

$$A\pi_R^* u \in I^m(X^2, \mathrm{Diag}; \pi_L^* F \otimes \Lambda_R). \tag{L7.16}$$

Then the 'action' of the operator is defined by integrating out the right, y, variables to get

$$Au = (\pi_L)_*(A \cdot \pi_R^* u). \tag{L7.17}$$

The push-forward theorem (using the freedom to choose the normal fibration) shows that this is an element of $\mathcal{C}^\infty(X;F)$.

Essentially the same argument works for composition of $B \in \Psi^{m'}(X;F,G)$ and $A \in \Psi^{-\infty}(X;E,F)$ except that we have three factors of X to worry about. However the right-most fact here can be viewed as a parameter space. The composition looks like

$$\int_X B(x,y)A(y,z)\text{`}dy\text{'} \tag{L7.18}$$

(where I have written 'dy' because the measure is already part of B) and we may interpret this as in (L7.17) by writing it

$$AB = (\pi_C)_*(\pi_S^* A \cdot \pi_F^* B). \tag{L7.19}$$

Here there are three projections from X^3 to X^2

(L7.20)

$$\pi_F : X^3 \ni (x,y,z) \longrightarrow (y,z) \in X^2,$$
$$\pi_C : X^3 \ni (x,y,z) \longrightarrow (x,z) \in X^2 \text{ and } \pi_S : X^3 \ni (x,y,z) \longrightarrow (x,y) \in X^2.$$

The first one drops the left variable, the second the middle variable and the last the right-most variable. The labels as supposed to correspond to the action of operators, as in $C = BA$, so A is the 'first' operator (in action) and corresponds to π_F, B is the 'second' operator and corresponds to π_S whereas C is the 'composite' operator and corresponds to π_C in (L7.18) and (L7.19); so you can think of this as the 'composite' projection or the 'central' projection.

Since these maps are smooth, $\pi_F^* A \in \mathcal{C}^\infty(X^3; \pi_M^* F \otimes \pi_R^* E')$ where

$$\begin{aligned} \pi_R &: X^3 \ni (x,y,z) \longrightarrow z \in X \\ \pi_M &: X^3 \ni (x,y,z) \longrightarrow y \in X \text{ and} \\ \pi_L &: X^3 \ni (x,y,z) \longrightarrow x \in X \end{aligned} \tag{L7.21}$$

are the three projections onto a single factor of X (corresponding to 'right', 'middle' and 'left'. We are using these projections mainly to pull bundles back. The pull-back theorem for conormal distributions proved above applies to show that

$$\pi_S^* A \in I^{m-\frac14 \dim X}(X^3, \pi_S^{-1}\operatorname{Diag}; \pi_L^* G \otimes \pi_M^* F \otimes \Omega_M). \tag{L7.22}$$

Thus the product in (L7.19) can be interpreted as an element

$$\pi_S^* A \cdot \pi_F^* B \in \mathcal{C}^\infty(X^2; \pi_L^* G \otimes \pi_R^* F \otimes \Omega_R) = \mathcal{C}^\infty(X^2; \operatorname{Hom}(E,G) \otimes \Omega_R). \tag{L7.23}$$

The global discussion of the composition when A is smoothing and B is pseudodifferential is similar. In fact it is not necessary to do it, since we know that the space of pseudodifferenetial operators is invariant under taking adjoints. Thus the discussion above then applies to $B^* A^*$ and this is $(AB)^*$.

Once we have taken care of the case where one of the factors is smoothing we can pass to the local setting. In fact, we can do that anyway. Thus if $\{U_i\}$ is an open cover of X we can decompose A and B into finite sums

$$A = \sum_{i,k} \psi_i A \Psi_k, \ B = \sum_{i',k'} \psi_{i'} B \Psi_{k'}. \tag{L7.24}$$

Then the composite decomposes into a big sum

$$(AB) = \sum_{i,k,i',k'} \psi_i A \Psi_k \psi_{i'} B \Psi_{k'}. \tag{L7.25}$$

Now, we have already discussed the case in which one of the factors is smoothing, which in particular covers the case where the support does not meet the diagonal. Let me prove this again by localization. Thus we can suppose that each element of the open cover $\{U_i\}$ is a coordinate neighbourhood over which the bundles E, F and G are trivial. The density bundle is trivialized by the coordinate density $|dx|$ so the kernels just become matrices of conormal distibutions with respect to the diagonal. The bundle composition is just matrix composition, so we are reduced to looking at each of the entries, just the composition of scalar kernels. In general there may have different coordinates in the various factors, but using Lemma 13 above we may assume that the middle patches, the left for B and the right for A, are the same. Now, if say the localized term on the right $A\Psi_k \psi_{i'} B \Psi_{k'}$ is smoothing, it can be regarded as a smooth map from $U_{k'}$ to smooth functions on $U_k = U_i$, using the fact that a smooth function on a product is the same as a smooth map from either factor into smooth functions on the other factor. Then applying $\psi_i A \psi_k$ on the left gives a smooth function on U_i, for each point in $U_{k'}$, where everything has compact support. The linearity and continuity of A means that it is a $\mathcal{C}^\infty$ map, so in fact this is a smooth map from $U_{k'}$ into $\mathcal{C}^\infty_c(U_i)$ and hence in fact an element of $\mathcal{C}^\infty_c(U_i \times U_{k'})$, i.e. the kernel of a smoothing operator. This gives the alternative proof of the composition formula where the right factor is smoothing, mentioned above. If the left factor is smoothing one can apply the discussion of adjoints as above.

Thus in the expansion of the product in (L7.25) we know that each term where one of the factors is smoothing is itself smoothing. Using a decomposition as in (L7.3) we arrive at (L7.5) and in fact by a similar argument we can see (changing the indexing) that if one of the terms in the product $A_l B_k$ is not smoothing then *both* factors have kernels supported in the product of a fixed element of the cover with itself, that is both have compact support in $U_i \times U_i$ for some i. This allows us to work in just one coordinate patch rather than two.

Thus, we are reduced to showing that the product AB in the case of compactly supported scalar pseudodifferential operators on $\mathbb{R}^n$. We choose to write B in right reduced form as in (L7.13) and A in left reduced form as in (L7.12)

$$\begin{gathered}\widehat{Bf}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} b_R(y,\xi) f(y) dy,\\ Af(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a_L(x,\xi) \hat f(\xi) d\xi,\ \forall\ f \in \mathcal{S}(\mathbb{R}^n).\end{gathered} \tag{L7.26}$$

Inserting the formula for B into that of A we see that the kernel of the composite is

$$AB = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x,\xi) b_R(y,\xi) d\xi. \tag{L7.27}$$

The product of the two symbols is a symbol itself, so this is almost of the form we expect, the inverse Fourier transform of a symbol. The problem is that it is not quite an inverse Fourier transform because both the variables x and y occur in the amplitude. However we have already effectively overcome this problem. Namely we can treat the dependence of the amplitude on, say, y as parameter and write (L7.27) in the form

$$AB(x,y) = \left((2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a_L(x,\xi) b_R(z,\xi) d\xi \right) \big|_{z=y}. \tag{L7.28}$$

Now the inverse Fourier transform gives a conormal distribution on $\mathbb{R}^{3n}$, with variables x, y, z, with respect to the submanifold $x = y$. Then restriction to $z = y$ is transversal to the submanifold so we deduce that the kernel is conormal and of order $m + m'$. Putting all the terms back together we deduce (L7.14) and also (L7.15). □

L7.4. Ellipticity again. Now, we can prove the same result as I showed last time for elliptic differential operators but in the more general setting of elliptic pseudodifferential operators.

THEOREM 3. *If $P \in \Psi^m(X; E, F)$ is elliptic, in the sense that $\sigma_m(P)$ is invertible at each point of S^*X, then there exists $Q \in \Psi^{-m}(X; F, E)$ such that*

$$QP = \mathrm{Id}_E - R_E,\ R_E \in \Psi^{-\infty},\ PQ = \mathrm{Id}_F - R_F,\ R_F \in \Psi^{-\infty}(X; F). \tag{L7.29}$$

PROOF. The proof is the same as for differential operators above, except that we use the composition formula from Proposition 13. Still, let me take the time to go through the proof again.

□

L7.5. Index problem. As a direct result of Theorem 3 that proof that an ellipti element $P \in \Psi^m(X; E, F)$ is Fredholm on $\mathcal{C}^\infty$ sections is reduced to the same statement for operators of the form $\mathrm{Id} + A$ with A smoothing. Namely we want to show that

$$\begin{gathered}\mathrm{Nul}(P) = \{u \in \mathcal{C}^\infty(X; E); Pu = 0\} \text{ is finite dimensional}\\ \mathrm{Ran}(P) = \{f \in \mathcal{C}^\infty(X; F);\ \exists\ u \in \mathcal{C}^\infty(X; E),\ Pu = f\} \text{ is closed and}\\ \mathcal{C}^\infty(X; F) = \mathrm{Ran}(P) + V,\ V \subset \mathcal{C}^\infty(X; F) \text{ finite dimensional.}\end{gathered} \tag{L7.30}$$

From (L7.29) we see that

$$\mathrm{Nul}(P) \subset \mathrm{Nul}(\mathrm{Id}_E - R_E) \text{ and } \mathrm{Ran}(P) \supset \mathrm{Ran}(\mathrm{Id}_F - R_F). \tag{L7.31}$$

So if $\mathrm{Nul}(\mathrm{Id}_E - R_E)$ is finite dimensional, so is $\mathrm{Nul}(P)$ and if $\mathrm{Ran}(\mathrm{Id}_F - R_F)$ is closed with finite codimension then so is $\mathrm{Ran}(P)$ (check the algebra here for yourself); the point being that for smoothing perturbations of the identity, this is always true. As noted before, the fact that the range is closed follows from the last condition, the existence of a finite dimensional complement. I include it to avoid confusion with the weaker condition that the closure of the range has finite codimension. I will talk extensively about smoothing operators, next time.

Now the index of P is by definition the integer

$$\mathrm{ind}(P) = \dim \mathrm{Nul}(P) - \dim\left(\mathcal{C}^\infty(X; F)\big/\mathrm{Ran}(P)\right), \tag{L7.32}$$

(although it might have been better if it had been defined with the opposite sign). The problem solved by the index theorem of Atiyah and Singer (in its simplest form) is the computation of the index in terms of the symbol of P, via a topological formula.

The question arises as to why this integer is interesting. Of course the fundamental reason is that it is something that does not occur in finite dimensions. For a finite dimensional matrix, the corresponding integer is the difference between row rank and collum rank so it just the difference of dimension of source and target vector spaces.

Practically the index solves the problem of 'perturbative invertibility', as I will show next week. Namely we can ask whether there exists a smoothing operator $R \in \Psi^{-\infty}(X; E, F)$ such that $P + R$ is inverible, meaning for present purposes that it is injective and surjective.

Proposition 14. *For any elliptic pseudodifferential operator $P \in \Psi^m(X; E, F)$ there exists $R \in \Psi^{-\infty}(X; E, F)$ such that $P + \epsilon R$ is invertible for small $\epsilon \neq 0$ if and only if* $\mathrm{ind}(P) = 0$.

To analyse the index I will need to detour a little into K-theory. Suppose Y is any compact manifold and E is any vector bundle over Y. Then consider the operators of the form $\mathrm{Id} + A$, $A \in \Psi^{-\infty}(X; E)$ as we have been doing, but now look at those which are invertible (as an operator on $\mathcal{C}^\infty(X : E)$. The inverse is automatically of the same form, so this is a group which I will denote $G^{-\infty}(Y; E)$. In fact it is an open subset of $\mathcal{C}^\infty(X^2; \mathrm{Hom}(E) \otimes \Omega_R)$ so has a well-defined topology. I will define K-theory directly through the definition of odd K-theory. Thus for any compact manifold X set

$$K^{-1}(X) = [X; G^{-\infty}(Y; E)] \tag{L7.33}$$

the set of (smooth) homotopy classes of smooth maps into $G^{-\infty}(Y;E)$. Of course it is implicit in this definition that the result is independent of the choice of Y or E.

7+. Addenda to Lecture 7