

Index for product-type families

Lecture 23: 8 December, 2005

It is probably a good thing that no one gave me notes of this lecture. This allows me to write something closer to what I should have said and try to undo some of the confusion I must have sown!

NB. At this stage, I have paid no serious attention to the coefficients in the expansion of the odd Chern character and correspondingly the eta forms. As a result you will find some discrepancies with the constants below — at some point I will track down these constants (and quite a few earlier ones!)

L23.1. Product-type Toeplitz algebra. I wanted to finish this course with an example of an index formula showing how the eta forms discussed last time enter as ‘regularized Chern forms’.¹ To make things reasonably simple² I will consider a product $M \times \mathbb{S}$ where M is an arbitrary compact manifold.³ To further simplify things I will consider operators on a fixed bundle and indeed a trivial one (this is not much of a restriction since one can always complement a bundle to be trivial, with the identity operator on the complement). Thus, the fibration is actually a product

$$(L23.1) \quad \begin{array}{c} M \times \mathbb{S} \\ \downarrow \pi \\ M. \end{array}$$

Rather than consider a general elliptic element $A \in \Psi_{\pi\text{-pt}}^{0,0}(M; \mathbb{C}^N)$ initially I will further restrict the problem by considering the Toeplitz algebra in this sense. Recall that the fibrewise operators

$$(L23.2) \quad \mathcal{C}^\infty(M; \Psi^0(\mathbb{S}; \mathbb{C}^N)) \subset \Psi_{\pi\text{-pt}}^{0,0}(M; \mathbb{C}^N).$$

In particular the Szegő projector, S , on \mathbb{S} lifts to an element of the product-type algebra. Thus we can consider the Toeplitz subalgebra of the product-type algebra

$$(L23.3) \quad \Psi_{\pi\text{-pt}, \mathcal{T}}^{0,0}(M; \mathbb{C}^N) = S\Psi_{\pi\text{-pt}}^{0,0}(M; \mathbb{C}^N)S.$$

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¹In fact I had hoped to do a reasonably general case, but there is still some work required for this.

²And let’s face it I had enough trouble with this as it is.

³If you are feeling energetic a natural generalization to understand would be that of a circle bundle, more precisely the circle bundle of a complex line bundle (hence oriented), over a compact manifold.

The symbol algebra for $\Psi_{\pi\text{-pt}}^{0,0}(M; \mathbb{C}^N)$ takes values in smooth maps in matrices on the blown-up cosphere bundle $[S^*(M \times \mathbb{S}), S^*M]$. In this case, because $S^*\mathbb{S}$ is just two copies of \mathbb{S} ,

$$(L23.4) \quad [S^*(M \times \mathbb{S}), S^*M] = S_+^*(M \times \mathbb{S}) \sqcup S_-^*(M \times \mathbb{S})$$

is just the disjoint of the two compact manifolds with boundary, consisting of the upper half-sphere bundle and the lower half-sphere bundle, separated by the blow-up rather than meeting at the equatorial sphere bundle. For a fibrewise family the symbol is the lift of the symbol of the family, so for the Szegő projector it is the identity on the upper half-sphere bundle and 0 on the lower half. Thus, symbol map gives a short exact sequence

$$(L23.5) \quad \Psi_{\pi\text{-pt}, \mathcal{T}}^{-1,0}(M; \mathbb{C}^N) \longrightarrow \Psi_{\pi\text{-pt}, \mathcal{T}}^{0,0}(M; \mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(S_+^*(M \times \mathbb{S}); M(N, \mathbb{C})).$$

The base family for a fibrewise family is just the family itself (lifted to the cosphere bundle of the base). So for the Toeplitz algebra in this product case we get a short exact sequence

$$(L23.6) \quad \Psi_{\pi\text{-pt}, \mathcal{T}}^{0,-1}(M; \mathbb{C}^N) \longrightarrow \Psi_{\pi\text{-pt}, \mathcal{T}}^{0,0}(M; \mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(S^*M; \Psi_{\mathcal{T}}^0(\mathbb{S}; M(N, \mathbb{C}))).$$

So, consider an elliptic element $A \in \Psi_{\pi\text{-pt}, \mathcal{T}}^{-1,0}(M; \mathbb{C}^N)$, meaning that the symbol $a = \sigma_0(A) \in \mathcal{C}^\infty(S_+^*(M \times \mathbb{S}); \text{GL}(N, \mathbb{C}))$ and the base family $\beta = \beta(A) \in \mathcal{C}^\infty(S^*M; \Psi_{\mathcal{T}}^0(\mathbb{S}; M(N, \mathbb{C})))$ are invertible, so $\beta^{-1} \in \mathcal{C}^\infty(S^*M; \Psi_{\mathcal{T}}^0(\mathbb{S}; M(N, \mathbb{C})))$. Under these conditions we may easily check that A defines a Fredholm operator⁴

$$(L23.7) \quad A : \mathcal{C}^\infty(M \times \mathbb{S}; \mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(M \times \mathbb{S}; \mathbb{C}^N)$$

and we wish to compute the index.

For an elliptic element of $\Psi^0(M \times \mathbb{S}; \mathbb{C}^N)$ the index is given by the formula of Atiyah and Singer. The assumptions we have made above mean that the formula simplifies. First, the relative Chern character reduces to the odd Chern character, since we are working on a trivial bundle and may take the trivial connection. Thus the topological image of the symbol is

$$(L23.8) \quad \text{Ch}(a) = \sum_k c_k \text{Tr}((a^{-1} da)^{2k+1})$$

then the index formula becomes

$$(L23.9) \quad \text{ind}(A) = \int_{S^*(M \times \mathbb{S})} \text{Td}(M) \wedge \text{Ch}(a).$$

In principle the Todd class of $M \times \mathbb{S}$ enters here, but this can easily be seen to reduce to the Todd class of M . Although I have not discussed Td in detail here, we may take it as the lift of a cohomology class on M , represented by some explicit deRham class, $\text{Td}(M) \in \mathcal{C}^\infty(M; \Lambda^{\text{evn}})$ pulled back to $S^*(M \times \mathbb{S})$.

So, what I want to show is how the eta forms enter in the corresponding formula for the elliptic elements of the Toeplitz algebra.

⁴Construct a parameterix!

THEOREM 14. For an elliptic element $A \in \Psi_{\pi\text{-pt}, \mathcal{T}}^{-1,0}(M; \mathbb{C}^N)$ the index is given by

$$(L23.10) \quad \text{ind}(A) = -\frac{1}{2} \int_{S^*M} \text{Td}(M) \wedge \eta(\beta) + \int_{S^*_+(M \times \mathbb{S})} \text{Td}(M) \wedge \text{Ch}(a),$$

$$\text{where } \eta(\beta) = \sum_k c_k \eta_{2k+1}(\beta) = \sum_k c_k \overline{\text{Tr}}((\beta^{-1} d\beta)^{2k+1})$$

and $\text{Ch}(a)$ is given by (L23.8).

L23.2. Variation formulæ. Let me deduce variation formulæ for the (odd) eta forms and the odd Chern character which follow from the computation, done earlier, of the exterior derivative in each case.

For the eta forms on the group $G_T^0(\mathbb{S}; \mathbb{C}^N) \hookrightarrow G_T^0(\mathbb{S}; \Psi^{-\infty}(Z; E))$ of invertible Toeplitz operators with values in $N \times N$ matrices

$$(L23.11) \quad \eta_{2k+1}(B) = \overline{\text{Tr}}((B^{-1} dB)^{2k+1})$$

we know, from Proposition 50, that

$$(L23.12) \quad d\eta_{2k+1}(B) = \sigma_0^*(\beta_{2k+2} + d\gamma_{2k+1})$$

where the forms on the right are both defined on the loop group $G_{(1)}^{-\infty}$ but in this case we have descended to $\text{GL}(N, \mathbb{C})$:

$$(L23.13) \quad \beta_{2k+2}(b) = -\frac{1}{2} \int_{\mathbb{S}} \text{tr}((b^{-1} db)^{2k+2} b^{-1} \partial_{\theta} b) d\theta,$$

$$\gamma_{2k+1}(b) = -\frac{1}{2} \int_{\mathbb{S}} \text{tr}((b^{-1} db)^{2k+1} b^{-1} \partial_{\theta} b) d\theta, \quad b \in \mathcal{C}^{\infty}(\mathbb{S}; \text{GL}(N, \mathbb{C})).$$

So, if we suppose that X is an oriented compact manifold without boundary and that

$$(L23.14) \quad F \in \mathcal{C}^{\infty}(X \times (0, 1)_t; G_T^0(\mathbb{S}; \mathbb{C}^N))$$

then we can consider the function

$$(L23.15) \quad h_1(t) = \int_X T \wedge F_t^* \eta_{2k+1}$$

where $T \in \mathcal{C}^{\infty}(X; \Lambda^{d-2k-1})$ is a fixed closed form. Certainly h_1 is a smooth function and

$$(L23.16) \quad h_1'(t) = \int_X T \wedge \frac{d}{dt} F_t^* \eta_{2k+1}.$$

Consider the pull back $F^* \eta_{2k+1}$ to $X \times (0, 1)$. It can be decomposed with respect to dt as

$$(L23.17) \quad F^* \eta_{2k+1} = \eta_{2k+1}(F, t) + dt \wedge \eta'_{2k}(F, t), \quad \eta_{2k+1}(F, t) = F_t^* \eta_{2k+1},$$

$$\eta'_{2k}(F, t) = \sum_{j=0}^{2k} (-1)^j F_t^* \overline{\text{Tr}}((F^{-1} d_X F)^j F^{-1} \frac{dF}{dt} (F^{-1} d_X F)^{2k-j})$$

where the coefficients are t -dependent forms on X . If we write out the formula for $dF^*\eta_{2k+1}$ obtained by pulling back (L23.12) we find

$$(L23.18) \quad dF^*\eta_{2k+1} = d_X\eta_{2k+1}(F, t) + dt \wedge \left(\frac{d}{dt}\eta_{2k+1}(F, t) - d_X\eta'_{2k}(F, t) \right) \\ = (F \circ \sigma)_0^*(\beta_{2k+2} + d\gamma_{2k+1}).$$

So, expanding the last pull-back with respect to dt gives

$$(L23.19) \quad \frac{d}{dt}\eta_{2k+1}(F, t) = d_X\eta'_{2k}(F, t) + \beta'_{2k+1}(f, t) + d_X\gamma'_{2k+1}(f, t)$$

where $f = \sigma \circ F$ is the symbol map for the family F and

(L23.20)

$$\beta'_{2k+1}(f, t) = -\frac{1}{2} \sum_{j=0}^{2k+1} (-1)^j \int_{\mathbb{S}} \text{Tr}((f^{-1}d_X f)^j (f^{-1}\frac{df}{dt})(f^{-1}d_X f)^{2k+1-j} f^{-1}\partial_\theta f) d\theta, \\ \gamma'_{2k+1}(f, t) = -\frac{1}{2} \sum_{j=0}^{2k} \int_{\mathbb{S}} \text{Tr}((f^{-1}d_X f)^j f^{-1}\frac{df}{dt}(f^{-1}d_X f)^{2k-j} f^{-1}\partial_\theta f) d\theta, \quad f = \sigma_0 \circ F.$$

So now, if we insert this formula into (L23.16) we find that the exact terms integrate to zero, so only β'_{2k+1} survives and

$$(L23.21) \quad h'_1(t) = \int_X T \wedge \beta'_{2k+1}(f, t)$$

with β'_{2k+1} given by (L23.20).

Next we make a similar computation for the Chern character. Suppose that Y is a compact manifold with boundary of dimension q , that $T' \in \mathcal{C}^\infty(Y; \Lambda^{q-2k-1})$ is closed and that $G : Y \times (0, 1) \rightarrow G^{-\infty}$ is smooth. Then consider the function

$$(L23.22) \quad h_2(t) = \int_Y T' \wedge G_t^* \beta_{2k+1}$$

where β_{2k+1} are the component forms for the odd Chern character,

$$(L23.23) \quad \beta_{2k+1} = \text{Tr}((a^{-1}da)^{2k+1}).$$

We know that $d\beta_{2k+1} = 0$ and from this we find

$$(L23.24) \quad \frac{d}{dt}G_t^* \beta_{2k+1} = d_Y \beta'_{2k}, \quad \beta'_{2k} = \text{Tr} \left((G^{-1}d_Y G)^{2k} G^{-1} \frac{dG}{dt} \right).$$

Using this and Stokes' theorem

$$(L23.25) \quad h'_2(t) = \int_Y T' \wedge d_Y \beta'_{2k} = \int_{\partial Y} i_{\partial Y}^* T' \wedge \beta'_{2k}.$$

In the application of these formulæ below, $\partial Y = X \times \mathbb{S}$. Thus the form β'_{2k} is pulled back to the product $X \times \mathbb{S}$. In this case, decomposing the total differential on ∂Y and carrying out the integral over the circle, first, shows that

(L23.26)

$$h'_2(t) = \sum_{j=0}^{2k-1} (-1)^j \int_{X \times \mathbb{S}} i_{\partial Y}^* T' \wedge \text{Tr} \left((g^{-1}d_X g)^j g^{-1}d_\theta g (g^{-1}d_X g)^{2k-1-j} g^{-1} \frac{dg}{dt} \right) d\theta,$$

$$g = G|_{\partial Y = X \times \mathbb{S}}.$$

Compare this to (L23.20), first increasing k to $k + 1$.

L23.3. Proof of Theorem 14. We first show that the right side of the formula (L23.10) is homotopy invariant. Thus, for an elliptic 1-parameter family $A_t \in \mathcal{C}^\infty((0, 1); \Psi_{\pi\text{-pt}, \mathcal{T}}^{-1,0}(M; \mathbb{C}^N))$ set

$$(L23.27) \quad I_1(t) = -\frac{1}{2} \int_{S^*M} \text{Td}(M) \wedge \eta(\beta), \quad I_2(t) = \int_{S_+^*(M \times \mathbb{S})} \text{Td}(M) \wedge \text{Ch}(a).$$

The discussion above allows us to compute the derivatives of these functions. Namely if we set

$$(L23.28) \quad b_t = \sigma_0(\beta(A_t)) = a_t|_{S^*M \times \mathbb{S}},$$

with equality being the consistency between the symbol and the base family, then from (L23.21), (L23.24) and (L23.25)⁵

$$(L23.29) \quad I_1'(t) + I_2'(t) = 0.$$

Thus the sum

$$(L23.30) \quad I = I_1(t) + I_2(t) \text{ is constant.}$$

So, to prove the index formula we may make a homotopy to some operator for which we can compute the index, in this case using the Atiyah-Singer theorem.⁶

Consider the symbol $a = \sigma_0(A) \in \mathcal{C}^\infty(S_+^*(M \times \mathbb{S}; \text{GL}(N, \mathbb{C})))$. This is a ball bundle, where the origin of each fibre can be taken as the fibre vertical, the point corresponding to

$$(L23.31) \quad M \times \mathbb{S} \ni (m, \theta) \longmapsto (m, \theta, 0, d\theta) \text{ that is } (0, 1) \in T_m^*M \times T_\theta^*\mathbb{S}.$$

Thus the fibres are contractible. This allows us to construct a smooth family of symbols

$$(L23.32) \quad \tilde{a} : [0, 1]_t \times S_+^*(M \times \mathbb{S}) \longrightarrow \text{GL}(N, \mathbb{C}), \quad \tilde{a}|_{t=0} = a, \quad \tilde{a}|_{t=1} = \pi^*(a|_{M \times \mathbb{S}}).$$

For instance, one can first smoothly deform the symbol so that it is fibre-constant near the ‘centre’ in (L23.31) and then radially translate the symbol to expand the constant region.

In particular the boundary symbols

$$(L23.33) \quad \tilde{b} = \tilde{a}|_{S^*M \times \mathbb{S} = \partial S_+^*(M \times \mathbb{S})} : [0, 1] \times S^*M \times \mathbb{S} \longrightarrow \text{GL}(N, \mathbb{C})$$

form a 1-parameter family of symbols of Toeplitz operators, in $\mathcal{C}^\infty(\mathbb{S}; \text{GL}(N, \mathbb{C}))$ parameterized by S^*M . The compatibility condition between symbols and base families shows that at $t = 0$, for the given operator, these are the symbols of an invertible family of Toeplitz operators. It follows from our earlier results about such families that we can lift to a family of invertible Toeplitz operators

$$(L23.34) \quad \tilde{\beta} : [0, 1] \times S^*M \longrightarrow \Psi_{\mathcal{T}}^0(\mathbb{S}; \mathbb{C}^N), \quad \tilde{\beta}^{-1} \in \mathcal{C}^\infty([0, 1] \times S^*M; \Psi_{\mathcal{T}}^0(\mathbb{S}; \mathbb{C}^N)), \\ \tilde{\beta}|_{t=0} = \beta(A).$$

⁵For the moment I am completely ignoring the constants – there is good reason to believe they work out correctly!

⁶In the lecture I was trying to run the homotopy in the wrong direction, which accounts for some of my discomfort.

Finally then we may select a smooth family $\tilde{A}_t \Psi_{\pi\text{-pt}, \mathcal{T}}^0(M \times \mathbb{S}; \mathbb{C}^N)$ with symbol \tilde{a} and base family $\tilde{\beta}$. This is therefore a family of (fully) elliptic operators and hence has constant index

$$(L23.35) \quad \text{ind}(A_t) \in \mathbb{Z} \text{ is smooth, hence constant.}$$

On the other hand we have also shown (L23.30), so it suffices to show that

$$(L23.36) \quad \text{ind}(\tilde{A}_1) = I(1),$$

which reduces to proving the index formula (L23.10) under the additional assumption that

$$(L23.37) \quad \sigma_0(A) \text{ is fibre constant on } S_+^*(M \times \mathbb{S}) \longrightarrow M \times \mathbb{S}.$$

This just means that $\sigma_0(A) = \sigma_0(L)$ where $L \in \mathcal{C}^\infty(M \times \mathbb{S}; \text{GL}(N, \mathbb{C}))$ is a bundle isomorphism for our trivial bundle. As an isomorphism L has vanishing index, so we may compose A with the inverse and strengthen (L23.37) further and suppose that

$$(L23.38) \quad \sigma_0(A) = \text{Id}.$$

Of course, we do have to check afterwards that the putative index formula behaves ‘correctly’ under multiplication by a bundle isomorphism.

Thus we are reduced to the case where the symbol is the identity. As we know, by a further (small) homotopy we may assume that the operator itself is of the form $\text{Id} + B$ where $B \in \Psi_{\pi\text{-pt}, \mathcal{T}}^{-\infty, 0}(M \times \mathbb{S}; \mathbb{C}^N)$, meaning in particular that its indicial family is

$$(L23.39) \quad \beta : S^*M \longrightarrow G_{\mathcal{T}}^{-\infty}(\mathbb{S}; \mathbb{C}^N),$$

so is of the form $\text{Id} + \text{smoothing}$ and everywhere invertible. This group (classifying for K^{-1}) is contained in our contractible Toeplitz group, so one might think that the whole thing could be contracted away and the index would then be zero. However this cannot be done without deforming the symbol to be non-trivial again. In fact this approach works perfectly well and reduces the problem to the Atiyah-Singer theorem on $M \times \mathbb{S}$.

Rather than do this I will go in the ‘opposite direction’. Namely we can further deform the base family, through invertibles of course, until it is a family of finite rank perturbations of the identity, just doing the deformation

$$(L23.40) \quad (1-t)\beta + t(\text{Id} - \pi_{(k)} + \pi_{(k)}\beta\pi_{(k)}), \quad k \text{ large}$$

where $\pi_{(k)}$ is projection onto the span of the first k terms in the Fourier expansion. It follows that the same deformation for the operator gives an elliptic family

$$(L23.41) \quad (1-t)A + t((\text{Id} - \pi_{(k)} + \pi_{(k)}A\pi_{(k)}))$$

so we are reduced to the case that A acts as the identity on the span of all Fourier coefficients greater than k (of course with arbitrary coefficients in $\mathcal{C}^\infty(M)$).

This corresponds to the inclusion $\Psi^0(M, \mathbb{C}^M) \hookrightarrow \Psi_{\pi\text{-pt}, \mathcal{T}}^{-\infty, 0}(M \times \mathbb{S}; \mathbb{C}^N)$ in which a pseudodifferential operator is lifted to a finite dimensional subbundle of $\mathcal{C}^\infty(M \times \mathbb{S}; \mathbb{C}^N)$ as a bundle over M . The symbol simply lifts to define the base symbol and the index to the index so we are finally reduced to the Atiyah-Singer theorem on M ,

$$(L23.42) \quad \text{ind}(A) = \int_M \text{Td}(M) \text{Ch}_{\text{odd}}(\sigma_0(A)), \quad A \in \Psi^0(M; \mathbb{C}^M) \text{ elliptic.}$$

By construction,⁷ the eta ‘character’ lifts the odd Chern character, so indeed (L23.42) reduces to, and hence implies, (L23.10) in this case.

Finally then it remains to check what happens to the formula under composition with a bundle isomorphism; I certainly did not do this in the lecture!

23+. Addenda to Lecture 23

23+.1. Composition with bundle isomorphisms.

23+.2. Non-Toeplitz extension. As a first simple generalization of the product-type index formula (L23.10) consider the case of a general elliptic element $A \in \Psi_{\pi\text{-pt}}^{0,0}(M \times \mathbb{S}; \mathbb{C}^N)$ for the product fibration (L23.1). The obvious generalization of (L23.10) is to include ‘both sides’ of the symbol and to extend the eta invariant to pseudodifferential operators on \mathbb{S} .

PROPOSITION 51. For any fully elliptic element $A \in \Psi_{\pi\text{-pt}}^{0,0}(M \times \mathbb{S}; \mathbb{C}^N)$

$$(23+.43) \quad \text{ind}(A) = -\frac{1}{2} \int_{S^*M} \text{Td}(M) \wedge \eta(\beta) + \sum_{\Sigma=\pm} \int_{S_{\Sigma}^*(M \times \mathbb{S})} \text{Td}(M) \wedge \text{Ch}(a), \quad \beta = \beta(A), \quad a = \sigma_0(A).$$

PROOF. For the moment I just assume that the right side of (23+.43) is homotopy invariant. This involves the extension of the analysis of the variation of the η forms to the full pseudodifferential calculus on the circle and is straightforward (there is an effective reversal of orientation between the two components of the cosphere bundle).

Granted this, it is enough to prove (23+.43) for some operator which is in the same path component of the full elliptic operators. Following the construction of homotopies above the operator may be deformed to one with symbol on in the positive half $S_+^*(M \times \mathbb{S})$ equal to that of a bundle isomorphism and then, composing with the inverse of this, to one with symbol equal to the identity on $S_+^*(M \times \mathbb{S})$. To proceed further we ‘separate’ the base family into the product of two Toeplitz families, one for the top and one for the bottom.

By a further small deformation we can assume that the full symbol of the base family is equal to the identity on the positive side of $S^*\mathbb{S}$, i.e. that the base family is of the form

$$(23+.44) \quad \beta(A) = \pi \text{Id} \pi + (\text{Id} - \pi) \alpha_- - (\text{Id} - \pi) + \gamma, \quad \gamma \in \mathcal{C}^\infty(B; \Psi^{-\infty}(\mathbb{S}; \mathbb{C}^N)), \quad \alpha_1 \in \mathcal{C}^\infty(B; \Psi_{-\mathcal{T}}^0(\mathbb{S}; \mathbb{C}^N))$$

and the whole operator is invertible. Since γ is a smoothing family

$$(23+.45) \quad \pi_{\leq k} \gamma \pi_{\leq k} \longrightarrow \gamma \in \mathcal{C}^\infty(B; \Psi^{-\infty}(\mathbb{S}; \mathbb{C}^N))$$

where $\pi_{\leq k}$ is projection onto all modes (including the negative ones) less than k . Thus when the norm of the difference in (23+.45) is sufficiently small we can replace γ by $\pi_{\leq k} \gamma \pi_{\leq k}$, and in fact A by

$$\pi_{\leq k} A \pi_{\leq k} + \text{Id} - \pi_{\leq k}$$

and so arrange that A itself acts as the identity on all Fourier modes $\exp(il\theta)$, $l > k$. Now, conjugating the whole operator by the bundle isomorphism $e^{i(k+1)\theta}$ gives a family

$$(23+.46) \quad A' = e^{-i(k+1)\theta} A e^{i(k+1)\theta} \in \mathcal{C}^\infty(B; \Psi_{\pi\text{-pt}, -\mathcal{T}}^{0,0}(M \times \mathbb{S}; \mathbb{C}^N))$$

⁷Modulo the constant chase

with values in the negative Toeplitz subalgebra, extended as the identity on the positive modes. Again from the invariance of the result under bundle isomorphisms it suffices to prove the formula for A' .

This reduces the problem to the previous case, with the orientation reversed. \square