CHAPTER 20

Chern character

Lecture 20: 29 November, 2005

We have the families index theorem in K-theory and now I want to discuss the image in cohomology.

Recall that in terms of K-theory we have shown that for any fibration of compact manifolds $Z \longrightarrow M \xrightarrow{\phi} B$ an elliptic element $A \in \Psi^m(M/B; E_+, E_-)$ can be stabilized by the addition of $A' \in \Psi^{-\infty}(M/B; E_+, E_-)$ so that the null spaces form a bundle and then

$$\operatorname{ind}(A) = [(\operatorname{null}(A + A'), \operatorname{null}((A + A')^*))] \in K^0(B) \tag{L20.1}$$

is the analytic index. As an element of the K-group it only depends on the image of the symbol of A in $K^0_{\mathrm{c}}(T^*(M/B))$.

Then for an embedding of the fibration

$$\begin{array}{ccc} M & \xrightarrow{\ e\ } & B \times \mathbb{S}^N \\ & {}_{\phi}\searrow \quad \swarrow_{\pi_1} & \\ & B & \end{array}$$

we can replace A with a family $P_A \in \Psi^0(B \times \mathbb{S}^N/B; G_+, G_-)$ having symbol given in terms of the Bott element and cut-offs

$$\begin{pmatrix} \chi_1 b & -\chi_2 a^* \\ \chi_2 a & \chi_2 b^* \end{pmatrix} \tag{L20.2}$$

in a collar neighbourhood of M and extended outside as the identity, with the property that $\operatorname{ind}(A) = \operatorname{ind}(P_A)$ in $K^0(B)$. This constructs a commutative diagram

$$\begin{array}{ccc} K^0_{\mathrm{c}}(T^*(M/B)) & \xrightarrow{\quad e_! \quad} & K^0_{\mathrm{c}}(\mathbb{R}^{2N} \times B) \\ & {}_{\operatorname{ind}}\searrow \quad \swarrow_{\operatorname{ind}} & \\ & K^0(B) & \end{array} \tag{L20.3}$$

where the index map on the right we 'understand completely' in the sense that it is given by repeated application of Bott periodicity, the index isomorphism for the Toeplitz calculus.

The traditional interpretation of (L20.3) is that the embedding construction defines the topological index, so the commutativity of (L20.3) is the equality of analytic and topological indexes. We can also think of it as an effective tool for

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computing the index. This will be more apparent in the generalization to product-type operators below.

L20.1. Review of Chern-Weil theory. Let $E \longrightarrow X$ be a complex vector bundle over a compact manifold. Then E always admits an affine connection which is to say a first order differential operator $\nabla \in \mathrm{Diff}^1(X; E, \Lambda^1 \otimes E)$:

$$\mathcal{C}^\infty(X; E) \xrightarrow{\nabla} \mathcal{C}^\infty(X; \Lambda^1 \otimes E) \tag{L20.4}$$

which has the property

$$\nabla(fu) = df \otimes u + f\nabla u \;\forall\; f \in \mathcal{C}^\infty(X),\; u \in \mathcal{C}^\infty(X; E). \tag{L20.5}$$

If ∇ is a connection on E and $a : E \longrightarrow \tilde{E}$ is a bundle isomorphism then $\tilde{\nabla}\tilde{u} = a\nabla(a^{-1}\tilde{u})$ is a connection on $\tilde{E}$. If $E = \mathbb{C}^N$ is trivial then d itself, acting on the coefficients, is a connection. If $\rho_i \in \mathcal{C}^\infty(X)$ is a partition of unity and ∇_i is a connection on E over an open set containing the support of ρ_i then

$$\nabla = \sum_i \rho_i \nabla_i \tag{L20.6}$$

is a connection on E. Combining these observations we see that any complex bundle does indeed admit a connection.

Any connection has a natural extension to a superconnection, which is to say an operator $\nabla \in \mathrm{Diff}^1(X; \Lambda^* X \otimes E)$ which satisfies

$$\nabla(\alpha \otimes u) = d\alpha \otimes u + (-1)^k \alpha \otimes \nabla u \;\forall\; \alpha \in \mathcal{C}^\infty(X; \Lambda^k),\; u \in \mathcal{C}^\infty(X; E),\; \forall\; k. \tag{L20.7}$$

The superconnection corresponding to an ordinary connection clearly satisfies the grading condition

$$\nabla \in \mathrm{Diff}^1(X; \Lambda^k \otimes E, \Lambda^{k+1} \otimes E) \;\forall\; k. \tag{L20.8}$$

The sign change corresponds to anticommuting ∇ past k wedge factors. Namely we can just insist on (L20.7) to get the superconnection with the connection on the right side; of course one still needs to check that the result is consistent. I will not distinguish between the connection ∇ and the superconnection it defines.

This allows us to define the curvature as the square of the connection which is always a bundle map

$$\mathcal{C}^\infty(X; \Lambda^2 \otimes \hom(E)) \ni \omega_\nabla = \nabla^2 \in \mathrm{Diff}^2(X; E, \Lambda^2 \otimes E). \tag{L20.9}$$

To see this, just observe that ∇^2 commutes with multiplication by any smooth function

$$\nabla^2(fu) = \nabla(df \otimes u + f\nabla u) = d^2 f \otimes u + (df \otimes \nabla u - df \otimes \nabla u) + f\nabla^2 u = f\nabla^2 u.$$

If $a : E \longmapsto \tilde{E}$ is a bundle isomorphism and $\tilde{\nabla} = a\nabla a^{-1}$ is the transformed connection then the curvature of $\tilde{\omega}$ of $\tilde{\nabla}$ is $a\omega_\nabla a^{-1}$. A connection on E induces a connection on the dual bundle E^* by demanding that

$$du^*(u) = \nabla^* u^*(u) + u^*(\nabla u) \;\forall\; u^* \in \mathcal{C}^\infty(X; E^*),\; u \in \mathcal{C}^\infty(X; E). \tag{L20.10}$$

The curvature of ∇^* is the transpose of the curvature of ∇. Similarly if E and F are bundles with connections ∇_E and ∇_F then the direct sum has the obvious connection $\nabla_E + \nabla_F$ with curvature $\omega_E + \omega_F$. Connections on E and F also induce a connection on $E \otimes F$ where for any sections

$$\nabla_{E\otimes F} u \otimes v = \nabla_E u \otimes v + u \otimes \nabla_F v. \tag{L20.11}$$

The curvature of this connection is easily computed

$$\omega_{E\otimes F} = \omega_E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \omega_F. \tag{L20.12}$$

Combining these two constructions we see that $\hom(E,F)$ also acquires a connection from connections on E and on F. Namely if we identify $\hom(E,F) = F \otimes E^*$ the connection is the tensor product of ∇_F and ∇_E^*. Alternatively one can see that the commutation formular

$$(\nabla a)u = \nabla_F(au) - a(\nabla_E)u,\ \forall\ u \in \mathcal{C}^\infty(X;E) \tag{L20.13}$$

defines the action of the connection on $a \in \mathcal{C}^\infty(X;\hom(E,F)$. Bianchi's identity, which comes from computing ∇_E^3 in two ways, then becomes the identity

$$\nabla_E \omega_E = 0 \tag{L20.14}$$

where ∇_E is also written for the (super) connection action on $\hom(E)$.

So, having defined the curvature of a connection we may define the Chern character form, or just the Chern character, of the bundle with connection as

$$\lambda_E = \operatorname{tr}\exp(\frac{i}{2\pi}\omega_E). \tag{L20.15}$$

The normalizing constant, $i/2\pi$, is put in for reasons of rationality (and is sometimes left out). To understand (L20.15) note first that the tensor product $\Lambda^* \otimes \hom(E)$ is a bundle of algebras over X. The product is just the tensor product of wedge and matrix products

$$(\alpha_p \otimes a_p)\cdot(\beta_p \otimes b_p) = \alpha_p \wedge \beta_p \otimes (a_p \circ b_p),\ \alpha_p,\ \beta_p \in \Lambda_p^*,\ a_p,\ b_p \in \hom(E_p). \tag{L20.16}$$

Then the exponential in (L20.15) is computed with respect to this product

$$\exp(\frac{i}{2\pi}\omega) = \mathrm{Id} + \sum_{k=1}^{\infty} \frac{i^k}{(2\pi)^k k!}\omega^k. \tag{L20.17}$$

Since ω takes values in 2-forms the sum is finite, since the power vanishes identically for $2k > \dim X$. Thus each term in the sum in (L20.17) is a smooth section of the bundle $\Lambda^{2k} \otimes \hom E$ over X. The trace functional, defined on $\hom(E)$ extends naturally to the tensor product

$$\operatorname{tr} : \mathcal{C}^\infty(X;\Lambda^j \otimes \hom E) \longrightarrow \mathcal{C}^\infty(X;\Lambda^j) \tag{L20.18}$$

and this is the meaning of (L20.15)

$$\lambda_E = r + \sum_{k=1}^{\infty} \frac{i^k}{(2\pi)^k k!} \operatorname{tr}(\omega^k) \in \mathcal{C}^\infty(X;\Lambda^{\mathrm{evn}}) \tag{L20.19}$$

where r is the rank of E (and the trace of the identity acting on it). Note that under a bundle isomorphism $a : E \longrightarrow \tilde{E}$ the form λ_E for a connection ∇ on E is the same as the form for the connection $a\nabla a^{-1}$ on $\tilde{E}$.

Lemma 33. *For any $a \in \mathcal{C}^\infty(X;\Lambda^k \otimes \hom E)$ and any connection*

$$d\operatorname{tr}(a) = \operatorname{tr}(\nabla a). \tag{L20.20}$$

Proof. We can cover X by open sets U_i over each of which E is trivial. Over these sets tr is given as the sum of the diagonal entries of the (form-valued) matrix a_i representing a. The connection on E over U_i can be compared to the trivial connection d and written $\nabla = d + \gamma_i$ where γ_i is a matrix valued in 1-forms (this follows directly from the definition of a connection); the action of the connection on a homomorphism, represented as a matrix, is then just

$$\nabla a = da + [\gamma_i, a]. \tag{L20.21}$$

Using a partition of unity ρ_i subordinate to the cover

$$\begin{aligned} d\,\mathrm{tr}(a) &= \sum_i d\,\mathrm{tr}(\rho_i a_i) = \sum_i \mathrm{tr}(d(\rho_i a_i)) \\ &= \sum_i \mathrm{tr}(d\rho_i a_{jk} + [\gamma_i, \rho_i a_i]) = \sum_i \mathrm{tr}(\nabla_E \rho_i a) = \mathrm{tr}(\nabla a). \end{aligned} \tag{L20.22}$$

□

From this lemma if follows immediately that

$$d\lambda_E = \mathrm{tr}\left(\nabla_E \exp(\frac{i}{2\pi}\omega_E)\right) = 0 \tag{L20.23}$$

since $\nabla\,\mathrm{Id} = 0$ and ∇ (acting on homomorphism) distributes over products, so $\nabla_E \omega_E^k = 0$ for every k.

Proposition 45. *The cohomology class of λ_E in $H^{evn}(X;\mathbb{C})$ is independent of the connection on E used to define it and this defines a group homomorphism*

$$\mathrm{Ch} : K^0(X) \longrightarrow H^{evn}(X;\mathbb{C}), \tag{L20.24}$$

the Chern character.

Proof. To show the independence of the choice of connection we use a standard 'transgression' analysis. Suppose ∇ and ∇' are two connections on E. Then

$$\tilde{\nabla} = (1-t)\nabla + t\nabla' + \partial_t dt \tag{L20.25}$$

is a connection on the bundle $\pi^* E$ over $[0,1] \times X$ where π is the projection onto X. Let $\tilde{\lambda}$ be the Chern form of this connection. From the discussion above, $\tilde{\lambda}$ is a (sum of) closed form(s) on $[0,1] \times X$ so, decomposing in terms of t-dependent forms on X

$$\tilde{\lambda} = \lambda' + dt \wedge \mu,\ d\tilde{\lambda} = 0 \Longrightarrow \partial_t \lambda = d\mu. \tag{L20.26}$$

Now, the Chern forms of ∇ and ∇' are respectively $\lambda'\big| + t = 0$ and $\lambda'\big|_{t=1}$ which are cohomologous since

$$\lambda'\big|_{t=1} - \lambda'\big|_{t=0} = \int_0^1 \partial_t \lambda' dt = d\int_0^1 \mu dt. \tag{L20.27}$$

For the direct sum of two bundle $E \oplus F$ we can choose a direct sum connection. Then, as noted above, the curvature is the sum of the curvatures, the one acting on E the other on F. As such an product of the two curvatures vanishes, so

$$\exp(\frac{i}{2\pi}(\omega_E + \omega_F)) = \exp(\frac{i}{2\pi}\omega_E) + \exp(\frac{i}{2\pi}\omega_F) \Longrightarrow \lambda_{E\oplus F} = \lambda_E + \lambda_F. \tag{L20.28}$$

This shows that the map

$$K^0(X) \ni [(E_+, E_-)] \longrightarrow [\lambda_{E_+} - \lambda_{E_-}] \in H^{\mathrm{evn}}(X;\mathbb{C}) \tag{L20.29}$$

is well-defined, since it is invariant under the addition of the same bundle to both E_+ and E_- and under bundle isomorphisms. □

As well as being an Abelian group, $K^0(X)$ is a ring with the product being induced by the tensor product of bundles. In fact we have already used this in the construction of P_A above. Suppose that $\mathbb{E}$ and $\mathbb{F}$ are superbundles, just $\mathbb{Z}_2$-graded bundles, $\mathbb{E} = (E_+, E_-)$ and $\mathbb{F} = (F_+, F_-)$. Then the graded tensor product is the bundle $\mathbb{G} = (G_+, G_-)$ where

$$G_+ = (E_+ \otimes F_+) \oplus (E_- \otimes F_-),\ G_- = (E_+ \otimes F_-) \oplus (E_- \otimes F_+).$$

It is straightforward to check that the equivalence class of $\mathbb{E} \otimes \mathbb{F}$ is determined by the classes of $\mathbb{E}$ and $\mathbb{F}$ and that this product on $K^0(X)$ is Abelian.

Since we know that for the tensor product of connections on E and F the curvature of $E \otimes F$ is $\omega_E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \omega_F$ it follows directly that

$$\lambda_{E\otimes F} = \lambda_E \wedge \lambda_F. \tag{L20.30}$$

Using the formula for direct sums as well and setting $\lambda_{\mathbb{E}} = \lambda_{E_+} - \lambda_{E_-}$ it follows that

$$\lambda_{\mathbb{E}\otimes\mathbb{F}} = \lambda_{\mathbb{E}} \wedge \lambda_{\mathbb{F}} \tag{L20.31}$$

as well. Thus in fact the Chern character is a multiplicative map

$$\mathrm{Ch} : K^0(X) \longrightarrow H^{\mathrm{evn}}(X;\mathbb{C}),\ \mathrm{Ch}(a\cdot b) = \mathrm{Ch}(a) \wedge \mathrm{Ch}(b)\ \forall\ a, b \in K^0(X) \tag{L20.32}$$

where the wedge product in deRham theory is the usual cup product. With a little more care it can be seen that Ch is well defined mapping into rational cohomology. It is important to know

Theorem 11. *(Atiyah-Hirzebruch) After tensoring with $\mathbb{C}$ the Chern character becomes and isomorphism*

$$K^0(X) \otimes \mathbb{C} \xrightarrow{\simeq} H^{evn}(X;\mathbb{C}). \tag{L20.33}$$

I will not discuss the proof of this (nor use it), although I hope that there is a treatment in the present spirit – at the moment I do not know one.

L20.2. Toeplitz families index. Recall that for elliptic families of Toeplitz operators, $A : B \longrightarrow \Psi^0_{\mathcal{T}}(\mathbb{S};\mathbb{C}^N)$ the families index theorem gives us Bott periodicity

$$\mathrm{ind} : K^{-2}(B) \longrightarrow K^0(B). \tag{L20.34}$$

Namely we can stabilize the symbol of the Toeplitz family

$$a = \sigma(A) \in \mathcal{C}^\infty(B \times \mathbb{S}; \mathrm{GL}(N,\mathbb{C})) \hookrightarrow \mathcal{C}^\infty(B \times \mathbb{S}; G^{-\infty})$$

and we can compose with the inverse of $\sigma(A)(b,1)$, as a bundle isomorphism over B, to normalize the symbol so that $A(b,1) = \mathrm{Id}$. This normalization does not change the index and a defines an element $[a] \in K^{-2}(B)$, as the homotopy class of $a : B \longrightarrow G^{-\infty}_{(1)}$, the pointed loop group. This gives the map (L20.34) which we know to be an isomorphism.

Thus the Chern character as discussed above on $K^0(B)$ induces a similar map from K^{-2} :

$$\begin{array}{ccc} K^{-2}(B) & \xrightarrow{\mathrm{ind}_{\mathcal{T}}} & K^0(B) \\ & \searrow^{\mathrm{Ch}} & \downarrow \mathrm{Ch} \\ & & H^{\mathrm{evn}}(B;\mathbb{C}). \end{array} \tag{L20.35}$$

I hope the notation is not be too confusing.

What we want is an explicit representative of this map in terms of $a \in \mathcal{C}^\infty(B \times S; \mathrm{GL}(N, \mathbb{C}))$, the Toeplitz symbol.

Proposition 46. *For any $a \in \mathcal{C}^\infty(B \times \mathbb{S}; \mathrm{GL}(N, \mathbb{C}))$ or $a \in \mathcal{C}^\infty(B \times \mathbb{S}; G^{-\infty})$ the Chern character is*

$$\mathrm{Ch}([a]) = \sum_{k=0}^{\infty} \frac{i^{k+1} k!}{(2\pi)^{k+1}(2k+1)!} \int_{\mathbb{S}} \mathrm{Tr}((a^{-1}da)^{2k+1}. \tag{L20.36}$$

The integrand in (L20.36) is a form on $B \times \mathbb{S}$ and the integral means push-forward. That is the form is $\alpha \wedge d\theta + \beta$ where α and β are θ-dependent forms on B and (L20.36) means the integral, with respect to θ, of α.

Proof. Recall that by stabilizing and extensively deforming a we reduced it to the form

$$a = \pi_-(b)e^{-i\theta} + \pi'_-(b) + \pi'_+(b) + \pi_+(b)e^{i\theta} \tag{L20.37}$$

where $\pi_\pm(b)$ are smooth families of projections on two trivial bundles $\mathbb{C}^{M_\pm}$ and $\pi'_\pm(b) = \mathrm{Id} - \pi_\pm(b)$ are their complementary projections. Thus, a is an elliptic symbol acting on $\mathbb{C}^M$, $M = M_- + M_+$. We know that if we quantize a to the family of Toeplitz operators

$$A = \pi_-(b)L + \pi'_-(b) + \pi'_+(b) + \pi_+(b)U \in \mathcal{C}^\infty(B; \Psi^0_{\mathcal{T}}(\mathbb{S}; \mathbb{C}^M)) \tag{L20.38}$$

then its null spaces for the bundle $E_- = \pi_-(b)\mathbb{C}^M_-$ realized in the constant functions on the circle and similarly for the adjoint, so the index is

$$\mathrm{ind}(a) = [\mathbb{E}] = [(\pi_-\mathbb{C}^{M_-}, \pi_+(b)\mathbb{C}^{M_+})] \in K^0(B). \tag{L20.39}$$

So, we need to compute the Chern forms for these two bundles, presented as the ranges of smooth families of projections on trivial bundles. For simiplicity of notation I will drop the signs for the moment and consider a subbundle $E = \pi(x)\mathbb{C}^N$ of a trivial bundle over a manifold X. Notice that this bundle is in no way special. So we need a connection on E and the obvious one is

$$\mathcal{C}^\infty(X; E) \ni u \longmapsto \pi(x)du \in \mathcal{C}^\infty(X; \Lambda^1 \otimes E). \tag{L20.40}$$

Here d acts on the coefficients. Now, we can write this operator as

$$\pi(x)d = d + (\mathrm{Id} - \pi(x))d = d + \pi'(x)d\pi(x) : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; \Lambda^1 \otimes E)$$

where d acts on the coefficients of $\pi(x)$ as a matrix. The superconnection takes the same form so the curvature is

(L20.41)
$$\omega_E u = (d+\pi'(x)d\pi(x))^2 u = d^2u + d(\pi'(d\pi)u) + \pi'(d\pi)du + \pi'(d\pi)\pi'(d\pi)u = (d\pi' \wedge d\pi)u$$

where I have used the identities that come from differentiating $\pi^2 = \pi$, namely $\pi'(d\pi)\pi' = \pi(d\pi)\pi = 0$. Here the wedge product is to be understood in terms of antisymmetrizing the value on the tangent space, not commutation of homomoprhisms. Since the curvature is acting on E we can write it out more fully as

$$\omega_E = -\pi(d\pi)(\pi')(d\pi)\pi \tag{L20.42}$$

and as already noted the product is in $\Lambda^* \hom$. Thus the Chern character form for these connections on the index bundle is

(L20.43) $$\mathrm{Ch}(\mathrm{ind}(a)) = \mathrm{tr}(\pi_-) - \mathrm{tr}(\pi_+)$$
$$+\sum_{k=1}^{\infty} \frac{i^k(-1)^k}{(2\pi)^k k!} \mathrm{tr}\left((\pi_-(d\pi_-)(\pi'_-)(d\pi_-)\pi_-)^k\right) - \sum_{k=1}^{\infty} \frac{i^k(-1)^k}{(2\pi)^k k!} \mathrm{tr}\left((\pi_+(d\pi_+)(\pi'_+)(d\pi_+)\pi_+)^k\right).$$

where the trace is on $\mathbb{C}^{M_\pm}$.

Now, we proceed to compute the correspondint terms in (L20.36). From (L20.37) we can compute the total differential, on $B \times \mathbb{S}$, which is what appears in (L20.36) but I will write here as $d' = d + d\theta\partial_\theta$ where d is the differential on B :

$$d'a = (-i\pi_- e^{-i\theta} + i\pi_+ e^{i\theta})d\theta + d\pi_- e^{-i\theta} - d\pi_- - d\pi_+ + d\pi_+ e^{i\theta}.$$

The inverse of a is simply $a(-\theta)$ and the composite is seen to be

(L20.44) $$a^{-1}d'a = (-i\pi_- + i\pi_+)d\theta$$
$$+ ((e^{-i\theta} - 1)\pi'_- + (1 - e^{i\theta})\pi_-)d\pi_- + ((e^{i\theta} - 1)\pi'_+ + (1 - e^{-i\theta})\pi_+)d\pi_+.$$

There is no interaction between the two terms so

(L20.45)
$$\mathrm{tr}\left((a^{-1}d'a)^{2k+1}\right) = \lambda'_k \wedge d\theta + \mu'_k,$$
$$\lambda'_k = -i(-1)^k(2k+1)(2 - e^{-i\theta} - e^{i\theta})^k \left(\pi_-(d\pi_-)\pi'_-(d\pi_-)\pi_-)^k - \pi_+(d\pi_+)\pi'_+(d\pi_+)\pi_+)^k\right).$$

Here the constant term in θ, with factor $d\theta$, which is what the integral will pick out, is computed by noting that the first term in (L20.44) must arise from exactly one factor. There are $2k + 1$ choices for this and commuting the chosen factor to the front results in no overall change of sign. Since $\pi_- d\pi_- \pi_- = 0$ the next factor can be replaced by the π'_- part, and so on alternatively through the remaining factors. So we arrive at (L20.36) in the special case that a is given by (L20.37). So, to compute the constant we need to evaluate

(L20.46) $$\int_0^{2\pi} (2 - e^{-i\theta} - e^{i\theta})^k d\theta = (-1)^k \int_0^{2\pi} (e^{-i\theta/2} - e^{i\theta/2})^{2k} d\theta = 2\pi \frac{(2k)!}{(k!)^2}.$$

However, from the earlier discussion of the forms in (L20.36), we know the cohomology classes to be stable under homotopy, and the forms are unchanged under stabilization by the identity. So in fact (L20.36) must always hold. □

Of course what we have computed is the Chern character of the index bundle for Toeplitz families.

Corollary 8. *If $A \in \mathcal{C}^\infty(B; \Psi^0_T(\mathbb{S}; \mathbb{C}^N)$ is an elliptic family of Toeplitz operators then the Chern character of its index bundle (in $K^0(B)$) is given by* (L20.36) *with $a = \sigma(A)$.*

Next time I will consider the relative Chern character, as a map from compactly supported K-theory. In particular we need to understand the map

(L20.47) $$\mathrm{Ch} : K_c(T^*(M/B)) \longrightarrow H^{\mathrm{evn}}_c(T^*(M/B); \mathbb{C})$$

since this is what appears in the cohomological version of the families index theorem

(L20.48) $$\mathrm{Ch}(\mathrm{ind}(A)) = \int \mathrm{Td}\,\mathrm{Ch}(\sigma(A)) \in H^{\mathrm{evn}}(B)$$

for an elliptic family $A \in \Psi^m(M/B; \mathbb{E})$. In fact we can already see that the map to cohomology of the base must be of this form, for some class Td which is independent of the operator. Next time I will identify Td .

20+. Addenda to Lecture 20