GRADUATE ANALYSIS I
FALL 2004

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1. Introduction

Here is the basic plan for the lectures:-

1. 7 lectures on measure and integration – Leading to Riesz representation theorem. (Reduced to 5 by popular demand.)
2. 7 lectures on distributions and Fourier transform – including Schwart space, Sobolev spaces and Sobolev embedding.
3. 7 lectures on differential operators with constant coefficients, fundamental solutions and hypoellipticity.
4. 4 lectures on operators, trace class, Hilbert-Schmid.

2. Lecture notes

Can be found at

`http://www-math.mit.edu/~rbm/18.155-F04.html`

as can photo-notes.

3. Problems

I anticipate that there will be 8 problems sets, each of which (with the possible exception of the first) will be substantial. Collaboration on these problems is permitted but solutions should be individually crafted.

3.1. Problem set 1.

3.1. Problem set 1. Due September 23, 1PM. (Sorry about saying September 16!) Problems 1-5, 10 (I am dropping 11 because I will not get far enough) from the notes. Here they are, with the wording changed a bit.

Problem 1. (Notes 1) Prove that u_+, defined by (1.10) is linear.

Problem 2. (Notes 2) Prove Lemma I.8.

Hint(s). All functions here are supposed to be continuous, I just don't bother to keep on saying it.

1. Recall, or check, that the local compactness of a metric space X means that for each point $x \in X$ there is an $\epsilon > 0$ such that the ball $\{y \in X; d(x,y) \le \delta\}$ is compact for $\delta \le \epsilon$.
2. First do the case $n = 1$, so $K \Subset U$ is a compact set in an open subset.
   1. (a) Given $\delta > 0$, use the local compactness of X, to cover K with a finite number of compact closed balls of radius at most δ.

  - (b) Deduce that if $\epsilon > 0$ is small enough then the set $\{x \in X; d(x,K) \le \epsilon\}$, where
$$d(x,K) = \inf_{y\in K} d(x,y),$$
is compact.
  - (c) Show that $d(x,K)$, for K compact, is continuous.
  - (d) Given $\epsilon > 0$ show that there is a continuous function $g_\epsilon : \mathbb{R} \longrightarrow [0,1]$ such that $g_\epsilon(t) = 1$ for $t \le \epsilon/2$ and $g_\epsilon(t) = 0$ for $t > 3\epsilon/4$.
  - (e) Show that $f = g_\epsilon \circ d(\cdot, K)$ satisfies the conditions for $n = 1$ if $\epsilon > 0$ is small enough.
- (3) Prove the general case by induction over n.
  - (a) In the general case, set $K' = K \cap U_1^{\complement}$ and show that the inductive hypothesis applies to K' and the U_j for $j > 1$; let f'_j, $j = 2, \dots, n$ be the functions supplied by the inductive assumption and put $f' = \sum_{j\ge2} f'_j$.
  - (b) Show that $K_1 = K \cap \{f' \le \frac{1}{2}\}$ is a compact subset of U_1.
  - (c) Using the case $n = 1$ construct a function F for K_1 and U_1.
  - (d) Use the case $n = 1$ again to find G such that $G = 1$ on K and $\operatorname{supp}(G) \Subset \{f' + F > \frac{1}{2}\}$.
  - (e) Make sense of the functions
$$f_1 = F\frac{G}{f' + F},\ f_j = f'_j\frac{G}{f' + F},\ j \ge 2$$
and show that they satisfies the inductive assumptions.

Problem 3. (Notes 3) (Easy) Show that σ-algebras are closed under countable intersections.

Problem 4. (Notes 4)(Easy) Show that if μ is a complete measure and $E \subset F$ where F is measurable and has measure 0 then $\mu(E) = 0$.

Problem 5. (Notes 5) Show that (in a locally compact metric space) compact subsets are measurable for any Borel measure. (This just means that compact sets are Borel sets if you follow through the tortuous terminology.)

Problem 6. (Notes 10) For the space $Y = \mathbb{N} = \{1, 2, \dots\} \subset \mathbb{R}$, describe $\mathcal{C}_0(Y)$ and guess a description of its dual in terms of sequences.

3.2. **Problem set 2: Due September 28.** From Notes: Problems 6, 11, 12, 13, 14.

Problem 1. Show that the smallest σ-algebra containing the sets
$$(a, \infty] \subset [-\infty, \infty]$$
for all $a \in \mathbb{R}$, is what is called above the 'Borel' σ-algebra on $[-\infty, \infty]$.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be any measure space (so μ is a measure on the σ-algebra $\mathcal{M}$ of subsets of X). Show that the set of equivalence classes of μ-integrable functions on X, with the equivalence relation
$$f_1 \equiv f_2 \Longleftrightarrow \mu(\{x \in X; f_1(x) \ne f_2(x)\}) = 0.$$
is a normed linear space with the usual linear structure and the norm given by
$$\|f\| = \int_X |f| d\mu.$$

Problem 3. Let $(X, \mathcal{M})$ be a set with a σ-algebra. Let $\mu : \mathcal{M} \to \mathbb{R}$ be a finite measure in the sense that $\mu(\phi) = 0$ and for any $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ with $E_i \cap E_j = \phi$ for $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \tag{1}$$

with the series on the right *always* absolutely convergenct (i.e., this is part of the requirement on μ). Define

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \tag{2}$$

for $E \in \mathcal{M}$, with the supremum over *all* measurable decompositions $E = \bigcup_{i=1}^{\infty} E_i$ with the E_i disjoint. Show that $|\mu|$ is a finite, positive measure.

Hint 1. You must show that $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(A_i)$ if $\bigcup_i A_i = E$, $A_i \in \mathcal{M}$ being disjoint. Observe that if $A_j = \bigcup_l A_{jl}$ is a measurable decomposition of A_j then together the A_{jl} give a decomposition of E. Similarly, if $E = \bigcup_j E_j$ is any such decomposition of E then $A_{jl} = A_j \cap E_l$ gives such a decomposition of A_j.

Hint 2. See [1] p. 117!

Problem 4. (Hahn Decomposition)

With assumptions as in Problem 3:

1. Show that $\mu_+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu_- = \frac{1}{2}(|\mu| - \mu)$ are positive measures, $\mu = \mu_+ - \mu_-$. Conclude that the definition of a measure in the notes based on (4.17) is the *same* as that in Problem 3.
2. Show that $\mu_\pm$ so constructed are orthogonal in the sense that there is a set $E \in \mathcal{M}$ such that $\mu_-(E) = 0$, $\mu_+(X \setminus E) = 0$.

   Hint. Use the definition of $|\mu|$ to show that for any $F \in \mathcal{M}$ and any $\epsilon > 0$ there is a subset $F' \in \mathcal{M}$, $F' \subset F$ such that $\mu_+(F') \geq \mu_+(F) - \epsilon$ and $\mu_-(F') \leq \epsilon$. Given $\delta > 0$ apply this result repeatedly (say with $\epsilon = 2^{-n}\delta$) to find a decreasing sequence of sets $F_1 = X$, $F_n \in \mathcal{M}$, $F_{n+1} \subset F_n$ such that $\mu_+(F_n) \geq \mu_+(F_{n-1}) - 2^{-n}\delta$ and $\mu_-(F_n) \leq 2^{-n}\delta$. Conclude that $G = \bigcap_n F_n$ has $\mu_+(G) \geq \mu_+(X) - \delta$ and $\mu_-(G) = 0$. Now let G_m be chosen this way with $\delta = 1/m$. Show that $E = \bigcup_m G_m$ is as required.

Problem 5. Now suppose that μ is a finite, positive Radon measure on a locally compact metric space X (meaning a finite positive Borel measure outer regular on Borel sets and inner regular on open sets). Show that μ is inner regular on all Borel sets and hence, given $\epsilon > 0$ and $E \in \mathcal{B}(X)$ there exist sets $K \subset E \subset U$ with K compact and U open such that $\mu(K) \geq \mu(E) - \epsilon$, $\mu(E) \geq \mu(U) - \epsilon$.

Hint. First take U open, then use *its* inner regularity to find K with $K' \Subset U$ and $\mu(K') \geq \mu(U) - \epsilon/2$. How big is $\mu(E \backslash K')$? Find $V \supset K' \backslash E$ with V open and look at $K = K' \backslash V$.

3.3. Problem set 3: Due October 5. From notes: Problems 16, 17, 18, 21, 24

Problem 1. Let $\| \; \|$ be a norm on a vector space V. Show that $\|u\| = (u, u)^{1/2}$ for an inner product satisfying the conditions of a pre-Hilbert space if and only if the parallelogram law holds for every pair $u, v \in V$.

Hint (From Dimitri Kountourogiannis)

If $\|\cdot\|$ comes from an inner product, then it must satisfy the polarisation identity:

$$(x,y) = 1/4(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 - i\|x-iy\|^2)$$

i.e, the inner product is recoverable from the norm, so use the RHS (right hand side) to define an inner product on the vector space. You will need the paralellogram law to verify the additivity of the RHS. Note the polarization identity is a bit more transparent for real vector spaces. There we have

$$(x,y) = 1/2(\|x+y\|^2 - \|x-y\|^2)$$

both are easy to prove using $\|a\|^2 = (a,a)$.

Problem 2. Show (Rudin does it) that if $u : \mathbb{R}^n \to \mathbb{C}$ has continuous partial derivatives then it is differentiable at each point.

Problem 3. Consider the function $f(x) = \langle x\rangle^{-1} = (1+|x|^2)^{-1/2}$. Show that

$$\frac{\partial f}{\partial x_j} = l_j(x)\cdot\langle x\rangle^{-3}$$

with $l_j(x)$ a linear function. Conclude by *induction* that $\langle x\rangle^{-1} \in \mathcal{C}_0^k(\mathbb{R}^n)$ for *all* k.

Problem 4. Show that a linear *map*

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is continuous if and only if for *each* k there exist C and j such that if $|\alpha| \le k$ and $|\beta| \le k$

$$\sup\left|x^\alpha D^\beta T\varphi\right| \le C \sum_{|\alpha'|\le j,\ |\beta'|\le j} \sup_{\mathbb{R}^n}\left|x^{\alpha'}D^{\beta'}\varphi\right| \ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n). \tag{3}$$

Problem 5. Show that elements of $L^2(\mathbb{R}^n)$ are "continuous in the mean" i.e.,

$$\lim_{|t|\to 0}\int_{\mathbb{R}^n}|u(x+t)-u(x)|^2\,dx = 0\,. \tag{4}$$

3.4. **Problem set 4.** This was late going up, so anyone who really needs the time can wait until Thursday October 14 to hand in solutions. If you give them to me on Tuesday October 12 I should be able to mark them by October 14.

Problems 29, 40, 41 from the notes and the following additional problem:

[Note that there was a nasty typo in this question when I first put it up.]

Problem 1. If $u \in \mathcal{S}'(\mathbb{R})$ show that there exists $v \in \mathcal{S}'(\mathbb{R})$ satisfying

$$\frac{d}{dx}v = u. \tag{5}$$

Hint:- Show that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ has $\int_{\mathbb{R}}\phi(t)dt = 1$ then any $\psi \in \mathcal{S}(\mathbb{R})$ can be written in the form

$$\psi = c\phi + \frac{d}{dx}(A\psi), \tag{6}$$

Where

$$A : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \tag{7}$$

is a continuous linear operator.

3.5. **Problem set 5.** Due October 19, 2004. Problems from notes:- Numbers 27, 28, 29, 30, 31.

3.6. **Problem set 6.** Due November 4, 2004. Problems from notes:- Numbers 39, 42, 50, 63, 65.

3.7. **Problem set 6.** Due November 16, 2004. Problems from notes:- 61, 62, plus the following two problems also at the end of the problems in the notes as numbers 75, 76.

Restriction from Sobolev spaces. The Sobolev embedding theorem shows that a function in $H^m(\mathbb{R}^n)$, for $m > n/2$ is continuous – and hence can be restricted to a subspace of $\mathbb{R}^n$. In fact this works more generally. Show that there is a well defined *restriction map*

$$H^m(\mathbb{R}^n) \longrightarrow H^{m-\frac{1}{2}}(\mathbb{R}^n) \text{ if } m > \frac{1}{2} \tag{8}$$

with the following properties:

1. On $\mathcal{S}(\mathbb{R}^n)$ it is given by $u \longmapsto u(0, x')$, $x' \in \mathbb{R}^{n-1}$.
2. It is continuous and linear.

Hint: Use the usual method of finding a weak version of the map on smooth Schwartz functions; namely show that in terms of the Fourier transforms on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$

$$\widehat{u(0,\cdot)}(\xi') = (2\pi)^{-1} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi') d\xi_1, \ \forall \ \xi' \in \mathbb{R}^{n-1}. \tag{9}$$

Use Cauchy's inequality to show that this is continuous as a map on Sobolev spaces as indicated and then the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$ to conclude that the map is well-defined and unique.

Restriction by WF: From class we know that the product of two distributions, one with compact support, is defined provided they have no 'opposite' directions in their wavefront set:

$$(x, \omega) \in \mathrm{WF}(u) \Longrightarrow (x, -\omega) \notin \mathrm{WF}(v) \text{ then } uv \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n). \tag{10}$$

Show that this product has the property that $f(uv) = (fu)v = u(fv)$ if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Use this to define a restriction map to $x_1 = 0$ for distributions of compact support satisfying $((0, x'), (\omega_1, 0)) \notin \mathrm{WF}(u)$ as the product

$$u_0 = u\delta(x_1). \tag{11}$$

[Show that $u_0(f)$, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ only depends on $f(0, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$.

4. PROJECTS

To pass the course each student is required to carry out one of the projects which will be described starting in the second week. Since I may require these to be revised before they are acceptable I would suggest that you start rather early.

Here are some possible projects that I am thinking about:-

1. Radon-Nikodym theorem.
2. Kuiper's theorem: The group of unitary operators on a (separable infinite dimensional) Hilbert space is contractible.
3. Seeley's extension theorem.
4. Gibb's phenomenon.
5. Surjectivity of any non-trivial constant coefficient differential operator, $P : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$.

(6) Every elliptic differential operator with constant coefficients is surjective as a map on $\mathcal{C}^\infty(U)$, for any open set $U \subset \mathbb{R}^n$.
(7) Lidskii's theorem on trace class operators on $L^2(\mathbb{R}^n)$.

5. GRADES

The final grade will be based on the homework (50%) and the project (50%) there will be no tests or examinations.

6. LECTURE CONTENTS ANTICIPATED AND THEN AFTER DELIVERY

(1) September 9
Space of continuous functions, dual space, positivity.
(2) September 14
Outer measures and measures.
(3) September 16
Caratheodory's theorem.
(4) September 21
Measurable functions and the integral – including Lebesgue's theorem of dominated convergence.
(5) September 23
Riesz representation, L^p spaces and completeness, $L^2(X, \mu)$ and Hilbert space.
(6) September 28
Riesz representation for Hilbert space. Differentiability and Schwartz space of test functions.
(7) September 30
Properties of $\mathcal{S}(\mathbb{R}^n)$. Tempered distributions. Differentiation and differential operators. Fourier transform.
(8) October 5
Bump functions. Characterization of δ. Fourier inversion. Plancherel formula.
(9) October 7
Convolution and density. Fourier transform on $L^2(\mathbb{R}^n)$.
(10) October 12
Sobolev spaces and Sobolev embedding. Duality between Sobolev spaces.
(11) October 14
Schwartz representation theorem. Fundamental solution of $\partial_x + i\partial_y$. Support of a distribution; distributions of compact support (start).
(12) October 19
Compact supports. Convolution of distributions

$$\operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v)$$

if one, at least, has compact support. Fundamental solutions.
(13) October 21
Singular support, hypoellipticity, ellipticity – parametrices for elliptic operators.
(14) October 26
Fundamental solution of the heat operator, hypoellipticity, initial value problem. Homogeneity. The distributions $x_\pm^z$, $z \in \mathbb{C} \setminus (-\mathbb{N})$.

(15) October 28
Distributions supported at0. Homogeneous distributions of order $z \notin -\mathbb{N}$ on the line. Hadamard regularization. Cone supports.

(16) November 2
Singular support and products. Conic support and convolution.

(17) November 4
Wavefront set refines singular support. Scattering wavefront set. Product and wavefront set. The wave equation.

(18) November 9
Fundamental solution of the wave equation. Solution to the Cauchy problem.

(19) November 16
Operators and Schwartz' kernel theorem.

(20) November 18

(21) November 23

(22) November 30

(23) December 2

(24) December 7
Lidskii's theorem. Δ on the torus. Self-adjointness of $\Delta + V$. Spectral decomposition. Wave equation on the torus. Wave equation on torus with potential. $\Delta + V$ on $\mathbb{R}^n$ with $V \in \mathcal{C}_c(\mathbb{R}^n)$.

(25) December 9
Questions:- Trace as integral of the kernel over the diagonal. Microlocal analysis.

References

[1] W. Rudin, *Real and complex analysis*, third edition ed., McGraw-Hill, 1987.

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