

18. SOLUTIONS TO (SOME OF) THE PROBLEMS

Solution 18.1 (To Problem 10). (by Matjaž Konvalinka).

Since the topology on \mathbb{N} , inherited from \mathbb{R} , is discrete, a set is compact if and only if it is finite. If a sequence $\{x_n\}$ (i.e. a function $\mathbb{N} \rightarrow \mathbb{C}$) is in $\mathcal{C}_0(\mathbb{N})$ if and only if for any $\epsilon > 0$ there exists a compact (hence finite) set F_ϵ so that $|x_n| < \epsilon$ for any n not in F_ϵ . We can assume that $F_\epsilon = \{1, \dots, n_\epsilon\}$, which gives us the condition that $\{x_n\}$ is in $\mathcal{C}_0(\mathbb{N})$ if and only if it converges to 0. We denote this space by c_0 , and the supremum norm by $\|\cdot\|_0$. A sequence $\{x_n\}$ will be abbreviated to x .

Let l^1 denote the space of (real or complex) sequences x with a finite 1-norm

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|.$$

We can define pointwise summation and multiplication with scalars, and $(l^1, \|\cdot\|_1)$ is a normed (in fact Banach) space. Because the functional

$$y \mapsto \sum_{n=1}^{\infty} x_n y_n$$

is linear and bounded ($|\sum_{n=1}^{\infty} x_n y_n| \leq \sum_{n=1}^{\infty} |x_n| |y_n| \leq \|x\|_0 \|y\|_1$) by $\|x\|_0$, the mapping

$$\Phi: l^1 \mapsto c_0^*$$

defined by

$$x \mapsto \left(y \mapsto \sum_{n=1}^{\infty} x_n y_n \right)$$

is a (linear) well-defined mapping with norm at most 1. In fact, Φ is an isometry because if $|x_j| = \|x\|_0$ then $|\Phi(x)(e_j)| = 1$ where e_j is the j -th unit vector. We claim that Φ is also surjective (and hence an isometric isomorphism). If φ is a functional on c_0 let us denote $\varphi(e_j)$ by x_j . Then $\Phi(x)(y) = \sum_{n=1}^{\infty} \varphi(e_n) y_n = \sum_{n=1}^{\infty} \varphi(y_n e_n) = \varphi(y)$ (the last equality holds because $\sum_{n=1}^{\infty} y_n e_n$ converges to y in c_0 and φ is continuous with respect to the topology in c_0), so $\Phi(x) = \varphi$.

Solution 18.2 (To Problem 29). (Matjaž Konvalinka) Since

$$\begin{aligned} D_x H(\varphi) &= H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x) dx = \\ &= i \int_0^{\infty} \varphi'(x) dx = i(0 - \varphi(0)) = -i\delta(\varphi), \end{aligned}$$

we get $D_x H = C\delta$ for $C = -i$.

Solution 18.3 (To Problem 40). (Matjaž Konvalinka) Let us prove this in the case where $n = 1$. Define (for $b \neq 0$)

$$U(x) = u(b) - u(x) - (b-x)u'(x) - \dots - \frac{(b-x)^{k-1}}{(k-1)!}u^{(k-1)}(x);$$

then

$$U'(x) = -\frac{(b-x)^{k-1}}{(k-1)!}u^{(k)}(x).$$

For the continuously differentiable function $V(x) = U(x) - (1-x/b)^k U(0)$ we have $V(0) = V(b) = 0$, so by Rolle's theorem there exists ζ between 0 and b with

$$V'(\zeta) = U'(\zeta) + \frac{k(b-\zeta)^{k-1}}{b^k}U(0) = 0$$

Then

$$U(0) = -\frac{b^k}{k(b-\zeta)^{k-1}}U'(\zeta),$$

$$u(b) = u(0) + u'(0)b + \dots + \frac{u^{(k-1)}(0)}{(k-1)!}b^{k-1} + \frac{u^{(k)}(\zeta)}{k!}b^k.$$

The required decomposition is $u(x) = p(x) + v(x)$ for

$$p(x) = u(0) + u'(0)x + \frac{u''(0)}{2}x^2 + \dots + \frac{u^{(k-1)}(0)}{(k-1)!}x^{k-1} + \frac{u^{(k)}(0)}{k!}x^k,$$

$$v(x) = u(x) - p(x) = \frac{u^{(k)}(\zeta) - u^{(k)}(0)}{k!}x^k$$

for ζ between 0 and x , and since $u^{(k)}$ is continuous, $(u(x) - p(x))/x^k$ tends to 0 as x tends to 0.

The proof for general n is not much more difficult. Define the function $w_x: I \rightarrow \mathbb{R}$ by $w_x(t) = u(tx)$. Then w_x is k -times continuously differentiable,

$$w'_x(t) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(tx)x_i,$$

$$w''_x(t) = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(tx)x_i x_j,$$

$$w_x^{(l)}(t) = \sum_{l_1+l_2+\dots+l_i=l} \frac{l!}{l_1!l_2!\dots l_i!} \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \dots \partial x_i^{l_i}}(tx)x_1^{l_1}x_2^{l_2}\dots x_i^{l_i}$$

so by above $u(x) = w_x(1)$ is the sum of some polynomial p (od degree k), and we have

$$\frac{u(x) - p(x)}{|x|^k} = \frac{v_x(1)}{|x|^k} = \frac{w_x^{(k)}(\zeta_x) - w_x^{(k)}(0)}{k!|x|^k},$$

so it is bounded by a positive combination of terms of the form

$$\left| \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_i^{l_i}}(\zeta_x) - \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_i^{l_i}}(0) \right|$$

with $l_1 + \dots + l_i = k$ and $0 < \zeta_x < 1$. This tends to zero as $x \rightarrow 0$ because the derivative is continuous.

Solution 18.4 (Solution to Problem 41). (Matjž Konvalinka) Obviously the map $\mathcal{C}_0(\mathbb{B}^n) \rightarrow \mathcal{C}(\mathbb{B}^n)$ is injective (since it is just the inclusion map), and $f \in \mathcal{C}(\mathbb{B}^n)$ is in $\mathcal{C}_0(\mathbb{B}^n)$ if and only if it is zero on $\partial\mathbb{B}^n$, ie. if and only if $f|_{\mathbb{S}^{n-1}} = 0$. It remains to prove that any map g on \mathbb{S}^{n-1} is the restriction of a continuous function on \mathbb{B}^n . This is clear since

$$f(x) = \begin{cases} |x|g(x/|x|) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is well-defined, coincides with f on \mathbb{S}^{n-1} , and is continuous: if M is the maximum of $|g|$ on \mathbb{S}^{n-1} , and $\epsilon > 0$ is given, then $|f(x)| < \epsilon$ for $|x| < \epsilon/M$.

Solution 18.5. (partly Matjaž Konvalinka)

For any $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \varphi(x) dx \right| &\leq \int_{-\infty}^{\infty} |\varphi(x)| dx \leq \sup((1+|x|^2)|\varphi(x)|) \int_{-\infty}^{\infty} (1+|x|^2)^{-1} dx \\ &\leq C \sup((1+|x|^2)|\varphi(x)|). \end{aligned}$$

Thus $\mathcal{S}(\mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} \varphi dx$ is continous.

Now, choose $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \phi(x) dx = 1$. Then, for $\psi \in \mathcal{S}(\mathbb{R})$, set

$$(18.1) \quad A\psi(x) = \int_{-\infty}^x (\psi(t) - c(\psi)\phi(t)) dt, \quad c(\psi) = \int_{-\infty}^{\infty} \psi(s) ds.$$

Note that the assumption on ϕ means that

$$(18.2) \quad A\psi(x) = - \int_x^{\infty} (\psi(t) - c(\psi)\phi(t)) dt$$

Clearly $A\psi$ is smooth, and in fact it is a Schwartz function since

$$(18.3) \quad \frac{d}{dx}(A\psi(x)) = \psi(x) - c\phi(x) \in \mathcal{S}(\mathbb{R})$$

so it suffices to show that $x^k A\psi$ is bounded for any k as $|x| \rightarrow \pm\infty$. Since $\psi(t) - c\phi(t) \leq C_k t^{-k-1}$ in $t \geq 1$ it follows from (18.2) that

$$|x^k A\psi(x)| \leq C x^k \int_x^\infty t^{-k-1} dt \leq C', \quad k > 1, \quad \text{in } x > 1.$$

A similar estimate as $x \rightarrow -\infty$ follows from (18.1). Now, A is clearly linear, and it follows from the estimates above, including that on the integral, that for any k there exists C and j such that

$$\sup_{\alpha, \beta \leq k} |x^\alpha D^\beta A\psi| \leq C \sum_{\alpha', \beta' \leq j} \sup_{x \in \mathbb{R}} |x^{\alpha'} D^{\beta'} \psi|.$$

Finally then, given $u \in \mathcal{S}'(\mathbb{R})$ define $v(\psi) = -u(A\psi)$. From the continuity of A , $v \in \mathcal{S}(\mathbb{R})$ and from the definition of A , $A(\psi') = \psi$. Thus

$$dv/dx(\psi) = v(-\psi') = u(A\psi') = u(\psi) \implies \frac{dv}{dx} = u.$$

Solution 18.6. We have to prove that $\langle \xi \rangle^{m+m'} \widehat{u} \in L_2(\mathbb{R}^n)$, in other words, that

$$\int_{\mathbb{R}^n} \langle \xi \rangle^{2(m+m')} |\widehat{u}|^2 d\xi < \infty.$$

But that is true since

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \xi \rangle^{2(m+m')} |\widehat{u}|^2 d\xi &= \int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} (1 + \xi_1^2 + \dots + \xi_n^2)^m |\widehat{u}|^2 d\xi = \\ &= \int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} \left(\sum_{|\alpha| \leq m} C_\alpha \xi^{2\alpha} \right) |\widehat{u}|^2 d\xi = \sum_{|\alpha| \leq m} C_\alpha \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} \xi^{2\alpha} |\widehat{u}|^2 d\xi \right) \end{aligned}$$

and since $\langle \xi \rangle^{m'} \xi^\alpha \widehat{u} = \langle \xi \rangle^{m'} \widehat{D^\alpha u}$ is in $L^2(\mathbb{R}^n)$ (note that $u \in H^m(\mathbb{R}^n)$ follows from $D^\alpha u \in H^{m'}(\mathbb{R}^n)$, $|\alpha| \leq m$). The converse is also true since C_α in the formula above are strictly positive.

Solution 18.7. Take $v \in L^2(\mathbb{R}^n)$, and define subsets of \mathbb{R}^n by

$$E_0 = \{x : |x| \leq 1\},$$

$$E_i = \{x : |x| \geq 1, |x_i| = \max_j |x_j|\}.$$

Then obviously we have $1 = \sum_{i=0}^n \chi_{E_i}$ a.e., and $v = \sum_{j=0}^n v_j$ for $v_j = \chi_{E_j} v$. Then $\langle x \rangle$ is bounded by $\sqrt{2}$ on E_0 , and $\langle x \rangle v_0 \in L^2(\mathbb{R}^n)$; and on E_j , $1 \leq j \leq n$, we have

$$\frac{\langle x \rangle}{|x_j|} \leq \frac{(1 + n|x_j|^2)^{1/2}}{|x_j|} = (n + 1/|x_j|^2)^{1/2} \leq (2n)^{1/2},$$

so $\langle x \rangle v_j = x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$. But that means that $\langle x \rangle v = w_0 + \sum_{j=1}^n x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$.

If u is in $L^2(\mathbb{R}^n)$ then $\widehat{u} \in L^2(\mathbb{R}^n)$, and so there exist $w_0, \dots, w_n \in L^2(\mathbb{R}^n)$ so that

$$\langle \xi \rangle \widehat{u} = w_0 + \sum_{j=1}^n \xi_j w_j,$$

in other words

$$\widehat{u} = \widehat{u}_0 + \sum_{j=1}^n \xi_j \widehat{u}_j$$

where $\langle \xi \rangle \widehat{u}_j \in L^2(\mathbb{R}^n)$. Hence

$$u = u_0 + \sum_{j=1}^n D_j u_j$$

where $u_j \in H^1(\mathbb{R}^n)$.

Solution 18.8. Since

$$D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x) dx = i \int_0^{\infty} \varphi'(x) dx = i(0 - \varphi(0)) = -i\delta(\varphi),$$

we get $D_x H = C\delta$ for $C = -i$.

Solution 18.9. It is equivalent to ask when $\langle \xi \rangle^m \widehat{\delta}_0$ is in $L^2(\mathbb{R}^n)$. Since

$$\widehat{\delta}_0(\psi) = \delta_0(\widehat{\psi}) = \widehat{\psi}(0) = \int_{\mathbb{R}^n} \psi(x) dx = 1(\psi),$$

this is equivalent to finding m such that $\langle \xi \rangle^{2m}$ has a finite integral over \mathbb{R}^n . One option is to write $\langle \xi \rangle = (1 + r^2)^{1/2}$ in spherical coordinates, and to recall that the Jacobian of spherical coordinates in n dimensions has the form $r^{n-1} \Psi(\varphi_1, \dots, \varphi_{n-1})$, and so $\langle \xi \rangle^{2m}$ is integrable if and only if

$$\int_0^{\infty} \frac{r^{n-1}}{(1+r^2)^m} dr$$

converges. It is obvious that this is true if and only if $n-1-2m < -1$, ie. if and only if $m > n/2$.

Solution 18.10 (Solution to Problem31). We know that $\delta \in H^m(\mathbb{R}^n)$ for any $m < -n/1$. Thus is just because $\langle \xi \rangle^p \in L^2(\mathbb{R}^n)$ when $p < -n/2$. Now, divide \mathbb{R}^n into $n+1$ regions, as above, being $A_0 = \{\xi; |\xi| \leq 1\}$ and $A_i = \{\xi; |\xi_i| = \sup_j |\xi_j|, |\xi| \geq 1\}$. Let v_0 have Fourier transform χ_{A_0} and for $i = 1, \dots, n$, $v_i \in \mathcal{S}'(\mathbb{R}^n)$ have Fourier transforms $\xi_i^{-n-1} \chi_{A_i}$. Since $|\xi_i| > c\langle \xi \rangle$ on the support of \widehat{v}_i for each $i = 1, \dots, n$, each term

is in H^m for any $m < 1 + n/2$ so, by the Sobolev embedding theorem, each $v_i \in C_0^0(\mathbb{R}^n)$ and

$$(18.4) \quad 1 = \hat{v}_0 \sum_{i=1}^n \xi_i^{n+1} \hat{v}_i \implies \delta = v_0 + \sum_i D_i^{n+1} v_i.$$

How to see that this cannot be done with n or less derivatives? For the moment I do not have a proof of this, although I believe it is true. Notice that we are actually proving that δ can be written

$$(18.5) \quad \delta = \sum_{|\alpha| \leq n+1} D^\alpha u_\alpha, \quad u_\alpha \in H^{n/2}(\mathbb{R}^n).$$

This cannot be improved to n from $n + 1$ since this would mean that $\delta \in H^{-n/2}(\mathbb{R}^n)$, which it isn't. However, what I am asking is a little more subtle than this.