18. Solutions to (some of) the problems

Solution 18.1 (To Problem 10). (by Matjaž Konvalinka).

Since the topology on \( \mathbb{N} \), inherited from \( \mathbb{R} \), is discrete, a set is compact if and only if it is finite. If a sequence \( \{x_n\} \) i.e. a function \( \mathbb{N} \to \mathbb{C} \) is in \( C_0(\mathbb{N}) \) if and only if for any \( \epsilon > 0 \) there exists a compact (hence finite) set \( F_\epsilon \) so that \( |x_n| < \epsilon \) for any \( n \) not in \( F_\epsilon \). We can assume that \( F_\epsilon = \{1, \ldots, n_\epsilon\} \), which gives us the condition that \( \{x_n\} \) is in \( C_0(\mathbb{N}) \) if and only if it converges to 0. We denote this space by \( c_0 \), and the supremum norm by \( \| \cdot \|_0 \). A sequence \( \{x_n\} \) will be abbreviated to \( x \).

Let \( l^1 \) denote the space of (real or complex) sequences \( x \) with a finite 1-norm

\[
\|x\|_1 = \sum_{n=1}^{\infty} |x_n|.
\]

We can define pointwise summation and multiplication with scalars, and \( (l^1, \| \cdot \|_1) \) is a normed (in fact Banach) space. Because the functional

\[
y \mapsto \sum_{n=1}^{\infty} x_n y_n
\]

is linear and bounded \( (\|\sum_{n=1}^{\infty} x_n y_n\| \leq \sum_{n=1}^{\infty} |x_n| |y_n| \leq \|x\|_0 \|y\|_1) \) by \( \|x\|_0 \), the mapping

\[
\Phi: l^1 \to c_0^*
\]

defined by

\[
x \mapsto \left( y \mapsto \sum_{n=1}^{\infty} x_n y_n \right)
\]

is a (linear) well-defined mapping with norm at most 1. In fact, \( \Phi \) is an isometry because if \( |x_j| = \|x\|_0 \) then \( |\Phi(x)(e_j)| = 1 \) where \( e_j \) is the \( j \)-th unit vector. We claim that \( \Phi \) is also surjective (and hence an isometric isomorphism). If \( \varphi \) is a functional on \( c_0 \) let us denote \( \varphi(e_j) \) by \( x_j \). Then \( \Phi(x)(y) = \sum_{n=1}^{\infty} \varphi(e_n) y_n = \sum_{n=1}^{\infty} \varphi(y_n e_n) = \varphi(y) \) (the last equality holds because \( \sum_{n=1}^{\infty} y_n e_n \) converges to \( y \) in \( c_0 \) and \( \varphi \) is continuous with respect to the topology in \( c_0 \)), so \( \Phi(x) = \varphi \).

Solution 18.2 (To Problem 29). (Matjaž Konvalinka) Since

\[
D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x) \, dx =
\]

\[
i \int_{0}^{\infty} \varphi'(x) \, dx = i(0 - \varphi(0)) = -i\delta(\varphi),
\]

we get \( D_x H = C\delta \) for \( C = -i \).
Solution 18.3 (To Problem 40). (Matjaž Konvalinka) Let us prove this in the case where $n = 1$. Define (for $b \neq 0$)

$$U(x) = u(b) - u(x) - (b - x)u'(x) - \ldots - \frac{(b - x)^{k-1}}{(k - 1)!} u^{(k-1)}(x);$$

then

$$U'(x) = -\frac{(b - x)^{k-1}}{(k - 1)!} u^{(k)}(x).$$

For the continuously differentiable function $V(x) = U(x) - (1-x/b)^k U(0)$ we have $V(0) = V(b) = 0$, so by Rolle’s theorem there exists $\zeta$ between 0 and $b$ with

$$V'(\zeta) = U'(\zeta) + \frac{k(b - \zeta)^{k-1}}{b^k} U(0) = 0$$

Then

$$U(0) = -\frac{b^k}{k(b - \zeta)^{k-1}} U'(\zeta),$$

$$u(b) = u(0) + u'(0)b + \ldots + \frac{u^{(k-1)}(0)}{(k - 1)!} b^{k-1} + \frac{u^{(k)}(\zeta)}{k!} b^k.$$

The required decomposition is $u(x) = p(x) + v(x)$ for

$$p(x) = u(0) + u'(0)x + \frac{u''(0)}{2} x^2 + \ldots + \frac{u^{(k-1)}(0)}{(k - 1)!} x^{k-1} + \frac{u^{(k)}(0)}{k!} x^k,$$

$$v(x) = u(x) - p(x) = \frac{u^{(k)}(\zeta) - u^{(k)}(0)}{k!} x^k$$

for $\zeta$ between 0 and $x$, and since $u^{(k)}$ is continuous, $(u(x) - p(x))/x^k$ tends to 0 as $x$ tends to 0.

The proof for general $n$ is not much more difficult. Define the function $w_x: I \to \mathbb{R}$ by $w_x(t) = u(tx)$. Then $w_x$ is $k$-times continuously differentiable,

$$w'_x(t) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(tx)x_i,$$

$$w''_x(t) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(tx)x_ix_j,$$

$$w^{(l)}_x(t) = \sum_{l_1+l_2+\ldots+l_i=l} \frac{l!}{l_1! l_2! \ldots l_i!} \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \ldots \partial x_i^{l_i}}(tx)x_1^{l_1}x_2^{l_2} \ldots x_i^{l_i}$$
so by above \( u(x) = w_x(1) \) is the sum of some polynomial \( p \) (of degree \( k \)), and we have
\[
\frac{u(x) - p(x)}{|x|^k} = \frac{v_x(1)}{|x|^k} = \frac{w_x^{(k)}(\zeta_x) - w_x^{(k)}(0)}{k!|x|^k},
\]
so it is bounded by a positive combination of terms of the form
\[
\left| \frac{\partial^l u}{\partial x^{l_1}_1 \partial x^{l_2}_2 \cdots \partial x^{l_i}_i}(\zeta_x x) - \frac{\partial^l u}{\partial x^{l_1}_1 \partial x^{l_2}_2 \cdots \partial x^{l_i}_i}(0) \right|
\]
with \( l_1 + \cdots + l_i = k \) and \( 0 < \zeta_x < 1 \). This tends to zero as \( x \to 0 \) because the derivative is continuous.

**Solution 18.4** (Solution to Problem 41). (Matjž Konvalinka) Obviously the map \( C_0(\mathbb{B}^n) \to C(\mathbb{B}^n) \) is injective (since it is just the inclusion map), and \( f \in C(\mathbb{B}^n) \) is in \( C_0(\mathbb{B}^n) \) if and only if it is zero on \( \partial \mathbb{B}^n \), i.e. if and only if \( f|_{S^{n-1}} = 0 \). It remains to prove that any map \( g \) on \( S^{n-1} \) is the restriction of a continuous function on \( \mathbb{B}^n \). This is clear since
\[
f(x) = \begin{cases} 
|x|g(x/|x|) & x \neq 0 \\
0 & x = 0
\end{cases}
\]
is well-defined, coincides with \( f \) on \( S^{n-1} \), and is continuous: if \( M \) is the maximum of \( |g| \) on \( S^{n-1} \), and \( \epsilon > 0 \) is given, then \( |f(x)| < \epsilon \) for \( |x| < \epsilon/M \).

**Solution 18.5.** (partly Matjaž Konvalinka)

For any \( \varphi \in \mathcal{S}(\mathbb{R}) \) we have
\[
|\int_{-\infty}^{\infty} \varphi(x)dx| \leq \int_{-\infty}^{\infty} |\varphi(x)|dx \leq \sup(|1+|x|^2|\varphi(x)|)| \int_{-\infty}^{\infty} (1+|x|^2)^{-1}dx \leq C \sup(|1+|x|^2|\varphi(x)|)).
\]
Thus \( \mathcal{S}(\mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} \varphi dx \) is continuous.

Now, choose \( \phi \in C_c^\infty(\mathbb{R}) \) with \( \int_{\mathbb{R}} \phi(x)dx = 1 \). Then, for \( \psi \in \mathcal{S}(\mathbb{R}) \), set
\[
(18.1) \quad A\psi(x) = \int_{-\infty}^{x} (\psi(t) - c(\psi)\phi(t)) \ dt, \ c(\psi) = \int_{-\infty}^{\infty} \psi(s) \ ds.
\]
Note that the assumption on \( \phi \) means that
\[
(18.2) \quad A\psi(x) = -\int_{x}^{\infty} (\psi(t) - c(\psi)\phi(t)) \ dt
\]
Clearly \( A\psi \) is smooth, and in fact it is a Schwartz function since
\[
(18.3) \quad \frac{d}{dx}(A\psi(x)) = \psi(x) - c\phi(x) \in \mathcal{S}(\mathbb{R})
\]
so it suffices to show that $x^k A \psi$ is bounded for any $k$ as $|x| \to \pm \infty$. Since $\psi(t) - c \phi(t) \leq C_1 t^{-k-1}$ in $t \geq 1$ it follows from (18.2) that

$$|x^k A \psi(x)| \leq C x^k \int_x^\infty t^{-k-1} dt \leq C', \quad k > 1, \quad x > 1.$$ 

A similar estimate as $x \to -\infty$ follows from (18.1). Now, $A$ is clearly linear, and it follows from the estimates above, including that on the integral, that for any $k$ there exists $C$ and $j$ such that

$$\sup_{\alpha, \beta \leq k} |x^\alpha D^\beta A \psi| \leq C \sum_{\alpha', \beta' \leq j} \sup_{x \in \mathbb{R}} |x^{\alpha'} D^{\beta'} \psi|.$$ 

Finally then, given $u \in S'(\mathbb{R})$ define $v(\psi) = -u(A \psi)$. From the continuity of $A$, $v \in S(\mathbb{R})$ and from the definition of $A$, $A(\psi') = \psi$. Thus

$$dv/dx(\psi) = v(-\psi') = u(A \psi') = u(\psi) \implies \frac{dv}{dx} = u.$$ 

**Solution 18.6.** We have to prove that $\langle \xi \rangle^{m+m'} \hat{u} \in L^2(\mathbb{R}^n)$, in other words, that

$$\int_{\mathbb{R}^n} \langle \xi \rangle^{2(m+m')} |\hat{u}|^2 d\xi < \infty.$$ 

But that is true since

$$\int_{\mathbb{R}^n} \langle \xi \rangle^{2(m+m')} |\hat{u}|^2 d\xi = \int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} (1 + \xi_1^2 + \ldots + \xi_n^2)^m |\hat{u}|^2 d\xi =$$

$$= \int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} \left( \sum_{|\alpha| \leq m} C_\alpha \xi^{2\alpha} \right) |\hat{u}|^2 d\xi = \sum_{|\alpha| \leq m} C_\alpha \left( \int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} \xi^{2\alpha} |\hat{u}|^2 d\xi \right)$$

and since $\langle \xi \rangle^{m'} \xi^{\alpha} \hat{u} = \langle \xi \rangle^{m'} \hat{D^\alpha u}$ is in $L^2(\mathbb{R}^n)$ (note that $u \in H^m(\mathbb{R})$ follows from $D^\alpha u \in H^m(\mathbb{R})$, $|\alpha| \leq m$). The converse is also true since $C_\alpha$ in the formula above are strictly positive.

**Solution 18.7.** Take $v \in L^2(\mathbb{R}^n)$, and define subsets of $\mathbb{R}^n$ by

$$E_0 = \{ x : |x| \leq 1 \},$$

$$E_i = \{ x : |x| \geq 1, |x| = \max_j |x_j| \}.$$ 

Then obviously we have $1 = \sum_{i=0}^n \chi_{E_i}$ a.e., and $v = \sum_{j=0}^n v_j$ for $v_j = \chi_{E_j} v$. Then $\langle x \rangle$ is bounded by $\sqrt{2}$ on $E_0$, and $\langle x \rangle v_0 \in L^2(\mathbb{R}^n)$; and on $E_j$, $1 \leq j \leq n$, we have

$$\frac{\langle x \rangle}{|x|} \leq \frac{(1 + n |x_j|^2)^{1/2}}{|x_j|} = (n + 1 / |x_j|^2)^{1/2} \leq (2n)^{1/2},$$
so \( \langle x \rangle v_j = x_j w_j \) for \( w_j \in \mathcal{L}^2(\mathbb{R}^n) \). But that means that 
\( \langle x \rangle v = w_0 + \sum_{j=1}^n x_j w_j \) for \( w_j \in \mathcal{L}^2(\mathbb{R}^n) \).

If \( u \) is in \( \mathcal{L}^2(\mathbb{R}^n) \) then \( \hat{u} \in \mathcal{L}^2(\mathbb{R}^n) \), and so there exist \( w_0, \ldots, w_n \in \mathcal{L}^2(\mathbb{R}^n) \) so that

\[
\langle \xi \rangle \hat{u} = w_0 + \sum_{j=1}^n \xi_j w_j,
\]

in other words

\[
\hat{u} = \hat{u}_0 + \sum_{j=1}^n \xi_j \hat{u}_j
\]

where \( \langle \xi \rangle \hat{u}_j \in \mathcal{L}^2(\mathbb{R}^n) \). Hence

\[
u = u_0 + \sum_{j=1}^n D_j u
\]

where \( u_j \in \mathcal{H}^1(\mathbb{R}^n) \).

**Solution 18.8.** Since

\[
D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x) \, dx = i \int_{0}^{\infty} \varphi'(x) \, dx = i(0-\varphi(0)) = -i \delta(\varphi),
\]

we get \( D_x H = C \delta \) for \( C = -i \).

**Solution 18.9.** It is equivalent to ask when \( \langle \xi \rangle^m \hat{\delta}_0 \) is in \( \mathcal{L}^2(\mathbb{R}^n) \). Since

\[
\hat{\delta}_0(\psi) = \delta_0(\hat{\psi}) = \hat{\psi}(0) = \int_{\mathbb{R}^n} \psi(x) \, dx = 1(\psi),
\]

this is equivalent to finding \( m \) such that \( \langle \xi \rangle^{2m} \) has a finite integral over \( \mathbb{R}^n \). One option is to write \( \langle \xi \rangle = (1 + r^2)^{1/2} \) in spherical coordinates, and to recall that the Jacobian of spherical coordinates in \( n \) dimensions has the form \( r^{n-1} \Psi(\varphi_1, \ldots, \varphi_{n-1}) \), and so \( \langle \xi \rangle^{2m} \) is integrable if and only if

\[
\int_{0}^{\infty} \frac{r^{n-1}}{(1 + r^2)^m} \, dr
\]

converges. It is obvious that this is true if and only if \( n - 1 - 2m \leq -1 \), ie. if and only if \( m > n/2 \).

**Solution 18.10** (Solution to Problem 31). We know that \( \delta \in H^m(\mathbb{R}^n) \) for any \( m < -n/1 \). Thus is just because \( \langle \xi \rangle^p \in \mathcal{L}^2(\mathbb{R}^n) \) when \( p < -n/2 \). Now, divide \( \mathbb{R}^n \) into \( n+1 \) regions, as above, being \( A_0 = \{ \xi : |\xi| \leq 1 \} \) and \( A_i = \{ \xi : |\xi| = \sup_j |\xi_j|, |\xi| \geq 1 \} \). Let \( v_0 \) have Fourier transform \( \chi_{A_0} \) and for \( i = 1, \ldots, n, v_i \in \mathcal{S}(\mathbb{R}^n) \) have Fourier transforms \( \xi_i^{-n-1} \chi_{A_i} \). Since \( |\xi_i| > c \langle \xi \rangle \) on the support of \( \hat{v}_i \) for each \( i = 1, \ldots, n \), each term
is in $H^m$ for any $m < 1 + n/2$ so, by the Sobolev embedding theorem, each $v_i \in \mathcal{C}_0^0(\mathbb{R}^n)$ and

\begin{equation}
1 = \hat{v}_0 \sum_{i=1}^{n} \xi_i^{n+1} \hat{v}_i \implies \delta = v_0 + \sum_i D_i^{n+1} v_i.
\end{equation}

How to see that this cannot be done with $n$ or less derivatives? For the moment I do not have a proof of this, although I believe it is true. Notice that we are actually proving that $\delta$ can be written

\begin{equation}
\delta = \sum_{|\alpha| \leq n+1} D^\alpha u_\alpha, \ u_\alpha \in H^{n/2}(\mathbb{R}^n).
\end{equation}

This cannot be improved to $n$ from $n + 1$ since this would mean that $\delta \in H^{-n/2}(\mathbb{R}^n)$, which it isn’t. However, what I am asking is a little more subtle than this.