9. Fourier inversion

It is shown above that the Fourier transform satisfies the identity

\begin{equation}
\varphi(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \, d\xi \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).
\end{equation}

If \( y \in \mathbb{R}^n \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) set \( \psi(x) = \varphi(x + y) \). The translation-invariance of Lebesgue measure shows that

\[ \hat{\psi}(\xi) = \int e^{-ix\cdot\xi} \varphi(x + y) \, dx = e^{iy\cdot\xi} \hat{\varphi}(\xi). \]

Applied to \( \psi \) the inversion formula (9.1) becomes

\begin{equation}
\varphi(y) = \psi(0) = (2\pi)^{-n} \int \hat{\psi}(\xi) \, d\xi
= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi} \hat{\varphi}(\xi) \, d\xi.
\end{equation}

**Theorem 9.1.** Fourier transform \( \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) is an isomorphism with inverse

\begin{equation}
\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \mathcal{G}\psi(y) = (2\pi)^{-n} \int e^{iy\cdot\xi} \psi(\xi) \, d\xi.
\end{equation}

**Proof.** The identity (9.2) shows that \( \mathcal{F} \) is \( 1-1 \), i.e., injective, since we can remove \( \varphi \) from \( \hat{\varphi} \). Moreover,

\begin{equation}
\mathcal{G}\psi(y) = (2\pi)^{-n} \mathcal{F}\psi(-y)
\end{equation}

So \( \mathcal{G} \) is also a continuous linear map, \( \mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \). Indeed the argument above shows that \( \mathcal{G} \circ \mathcal{F} = \text{Id} \) and the same argument, with some changes of sign, shows that \( \mathcal{F} \circ \mathcal{G} = \text{Id} \). Thus \( \mathcal{F} \) and \( \mathcal{G} \) are isomorphisms.

**Lemma 9.2.** For all \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \), Paseval’s identity holds:

\begin{equation}
\int_{\mathbb{R}^n} \varphi\overline{\psi} \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}\overline{\hat{\psi}} \, d\xi.
\end{equation}
Proof. Using the inversion formula on $\varphi$,
\[
\int \varphi \overline{\psi} \, dx = (2\pi)^{-n} \int (e^{ix \cdot \xi} \varphi(\xi)) \overline{\psi(x)} \, dx
\]
\[
= (2\pi)^{-n} \int \varphi(\xi) \int e^{-ix \cdot \xi} \psi(x) \, dx \, d\xi
\]
\[
= (2\pi)^{-n} \int \varphi(\xi) \overline{\varphi(\xi)} \, d\xi.
\]
Here the integrals are absolutely convergent, justifying the exchange of orders.

Proposition 9.3. Fourier transform extends to an isomorphism
\[
(9.6) \quad \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).
\]
Proof. Setting $\varphi = \psi$ in (9.5) shows that
\[
(9.7) \quad \|\mathcal{F}\varphi\|_{L^2} = (2\pi)^{n/2} \|\varphi\|_{L^2}.
\]
In particular this proves, given the known density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, that $\mathcal{F}$ is an isomorphism, with inverse $\mathcal{G}$, as in (9.6).

Lemma 9.4. If $m \in \mathbb{N}$ is an integer, then
\[
(9.9) \quad u \in H^m(\mathbb{R}^n) \iff D^\alpha u \in L^2(\mathbb{R}^n) \quad \forall |\alpha| \leq m.
\]
Proof. By definition, $u \in H^m(\mathbb{R}^n)$ implies that $\langle \xi \rangle^{-m} \hat{u} \in L^2(\mathbb{R}^n)$. Since
\[
\widehat{D^\alpha u} = \xi^\alpha \hat{u}
\]
this certainly implies that $D^\alpha u \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq m$. Conversely if $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$ then $\xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$ and since
\[
\langle \xi \rangle^m \leq C_m \sum_{|\alpha| \leq m} |\xi^\alpha|.
\]
this in turn implies that $\langle \xi \rangle^m \hat{u} \in L^2(\mathbb{R}^n)$.
Now that we have considered the Fourier transform of Schwartz test functions we can use the usual method, of duality, to extend it to tempered distributions. If we set $\hat{\eta} = \psi$ and $\psi = G\hat{\psi} = G\hat{\eta}$ so

$$
\overline{\psi}(x) = (2\pi)^{-n} \int e^{-ix\xi} \overline{\psi}(\xi) d\xi
$$

$$
= (2\pi)^{-n} \int e^{-ix\xi} \eta(\xi) d\xi = (2\pi)^{-n} \hat{\eta}(x).
$$

Substituting in (9.5) we find that

$$
\int \varphi \hat{\eta} dx = \int \overline{\varphi} \eta d\xi.
$$

Now, recalling how we embed $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ we see that

(9.10) $u_{\varphi}(\eta) = u_{\varphi}(\hat{\eta}) \forall \eta \in \mathcal{S}(\mathbb{R}^n)$.

Definition 9.5. If $u \in \mathcal{S}'(\mathbb{R}^n)$ we define its Fourier transform by

(9.11) $\hat{u}(\varphi) = u(\hat{\varphi}) \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$.

As a composite map, $\hat{u} = u \cdot \mathcal{F}$, with each term continuous, $\hat{u}$ is continuous, i.e., $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.

Proposition 9.6. The definition (9.7) gives an isomorphism

$$
\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n), \quad \mathcal{F}u = \hat{u}
$$

satisfying the identities

(9.12) $\widehat{D^\alpha u} = \xi^\alpha u, \quad \overline{\widehat{x^\alpha u}} = (-1)^{|\alpha|} D^\alpha \hat{u}$.

Proof. Since $\hat{u} = u \circ \mathcal{F}$ and $\mathcal{G}$ is the 2-sided inverse of $\mathcal{F}$,

(9.13) $u = \hat{u} \circ \mathcal{G}$

gives the inverse to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$, showing it to be an isomorphism. The identities (9.12) follow from their counterparts on $\mathcal{S}(\mathbb{R}^n)$:

$$
\widehat{D^\alpha u}(\varphi) = D^\alpha u(\hat{\varphi}) = u((-1)^{|\alpha|} D^\alpha \hat{\varphi}) = u(\xi^\alpha \varphi) = \hat{u}((\xi^\alpha \varphi) = \xi^\alpha \hat{u}(\varphi) \forall \varphi \in \mathcal{S}(\mathbb{R}^n).
$$

We can also define Sobolev spaces of negative order:

(9.14) $H^m(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) ; \hat{u} \in (\xi)^{-m} L^2(\mathbb{R}^n) \}$.
Proposition 9.7. If \( m \leq 0 \) is an integer then \( u \in H^m(\mathbb{R}^n) \) if and only if it can be written in the form

\[
(9.15) \quad u = \sum_{|\alpha| \leq -m} D^\alpha v_\alpha, \quad v_\alpha \in L^2(\mathbb{R}^n).
\]

Proof. If \( u \in S'(\mathbb{R}^n) \) is of the form (9.15) then

\[
(9.16) \quad \hat{u} = \sum_{|\alpha| \leq -m} \xi^\alpha \hat{v}_\alpha \text{ with } \hat{v}_\alpha \in L^2(\mathbb{R}^n).
\]

Thus \( \langle \xi \rangle^m \hat{u} = \sum_{|\alpha| \leq -m} \xi^\alpha \langle \xi \rangle^m \hat{v}_\alpha \). Since all the factors \( \xi^\alpha \langle \xi \rangle^m \) are bounded, each term here is in \( L^2(\mathbb{R}^n) \), so \( \langle \xi \rangle^m \hat{u} \in L^2(\mathbb{R}^n) \) which is the definition, \( u \in \langle \xi \rangle^{-m} L^2(\mathbb{R}^n) \).

Conversely, suppose \( u \in H^m(\mathbb{R}^n) \), i.e., \( \langle \xi \rangle^m \hat{u} \in L^2(\mathbb{R}^n) \). The function

\[
\left( \sum_{|\alpha| \leq -m} \langle \xi \rangle^m \right) \cdot \langle \xi \rangle^m \in L^2(\mathbb{R}^n) \quad (m < 0)
\]

is bounded below by a positive constant. Thus

\[
v = \left( \sum_{|\alpha| \leq -m} \langle \xi \rangle^m \right)^{-1} \hat{u} \in L^2(\mathbb{R}^n).
\]

Each of the functions \( \hat{v}_\alpha = \text{sgn}(\xi^\alpha) \hat{v} \in L^2(\mathbb{R}^n) \) so the identity (9.16), and hence (9.15), follows with these choices.

Proposition 9.8. Each of the Sobolev spaces \( H^m(\mathbb{R}^n) \) is a Hilbert space with the norm and inner product

\[
\| u \|_{H^m} = \left( \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle \xi \rangle^{2m} d\xi \right)^{1/2},
\]

\[
\langle u, v \rangle = \int_{\mathbb{R}^n} \hat{u}(\xi) \overline{\hat{v}(\xi)} \langle \xi \rangle^{2m} d\xi.
\]

The Schwartz space \( S(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n) \) is dense for each \( m \) and the pairing

\[
H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n) \ni (u, u') \longmapsto ((u, u')) = \int_{\mathbb{R}^n} \hat{u}(\xi) \overline{\hat{u}'(\xi)} d\xi \in \mathbb{C}
\]

gives an identification \( (H^m(\mathbb{R}^n))' = H^{-m}(\mathbb{R}^n) \).
Proof. The Hilbert space property follows essentially directly from the definition (9.14) since $\langle \xi \rangle^{-m}L^2(\mathbb{R}^n)$ is a Hilbert space with the norm (9.17). Similarly the density of $\mathcal{S}$ in $H^m(\mathbb{R}^n)$ follows, since $\mathcal{S}(\mathbb{R}^n)$ dense in $L^2(\mathbb{R}^n)$ (Problem L11.P3) implies $\langle \xi \rangle^{-m}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ is dense in $\langle \xi \rangle^{-m}L^2(\mathbb{R}^n)$ and so, since $\mathcal{F}$ is an isomorphism in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$.

Finally observe that the pairing in (9.18) makes sense, since $\langle \xi \rangle^{-m}\hat{u}(\xi), \langle \xi \rangle^m\hat{u}'(\xi) \in L^2(\mathbb{R}^n)$ implies

$$\hat{u}(\xi)\hat{u}'(-\xi) \in L^1(\mathbb{R}^n).$$

Furthermore, by the self-duality of $L^2(\mathbb{R}^n)$ each continuous linear functional

$$U : H^m(\mathbb{R}^n) \rightarrow \mathbb{C}, U(u) \leq C\|u\|_{H^m}$$

can be written uniquely in the form

$$U(u) = ((u, u'))$$

for some $u' \in H^{-m}(\mathbb{R}^n)$.

Notice that if $u, u' \in \mathcal{S}(\mathbb{R}^n)$ then

$$((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x) dx.$$

This is always how we “pair” functions — it is the natural pairing on $L^2(\mathbb{R}^n)$. Thus in (9.18) what we have shown is that this pairing on test function

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (u, u') \mapsto ((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x) dx$$

extends by continuity to $H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n)$ (for each fixed $m$) when it identifies $H^{-m}(\mathbb{R}^n)$ as the dual of $H^m(\mathbb{R}^n)$. This was our ‘picture’ at the beginning.

For $m > 0$ the spaces $H^m(\mathbb{R}^n)$ represents elements of $L^2(\mathbb{R}^n)$ that have “$m$” derivatives in $L^2(\mathbb{R}^n)$. For $m < 0$ the elements are ?? of “up to $-m$” derivatives of $L^2$ functions. For integers this is precisely ??.