8. Convolution and density

We have defined an inclusion map
\[ S(\mathbb{R}^n) \ni \varphi \mapsto u_\varphi \in S'(\mathbb{R}^n), \quad u_\varphi(\psi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x) \, dx \quad \forall \psi \in S(\mathbb{R}^n). \]  

This allows us to ‘think of’ \( S(\mathbb{R}^n) \) as a subspace of \( S'(\mathbb{R}^n) \); that is we habitually identify \( u_\varphi \) with \( \varphi \). We can do this because we know (8.1) to be injective. We can extend the map (8.1) to include bigger spaces
\[ C^0_0(\mathbb{R}^n) \ni \varphi \mapsto u_\varphi \in S'(\mathbb{R}^n) \]
\[ L^p(\mathbb{R}^n) \ni \varphi \mapsto u_\varphi \in S'(\mathbb{R}^n) \]
\[ M(\mathbb{R}^n) \ni \mu \mapsto u_\mu \in S'(\mathbb{R}^n) \]
\[ u_\mu(\psi) = \int_{\mathbb{R}^n} \psi \, d\mu, \]

but we need to know that these maps are injective before we can forget about them.

We can see this using convolution. This is a sort of ‘product’ of functions. To begin with, suppose \( v \in C^0_0(\mathbb{R}^n) \) and \( \psi \in S(\mathbb{R}^n) \). We define a new function by ‘averaging \( v \) with respect to \( \psi \):’
\[ v * \psi(x) = \int_{\mathbb{R}^n} v(x - y)\psi(y) \, dy. \]

The integral converges by dominated convergence, namely \( \psi(y) \) is integrable and \( v \) is bounded,
\[ |v(x - y)\psi(y)| \leq ||v||_{C^0_0} |\psi(y)|. \]

We can use the same sort of estimates to show that \( v * \psi \) is continuous. Fix \( x \in \mathbb{R}^n \),
\[ v * \psi(x + x') - v * \psi(x) \]
\[ = \int_{\mathbb{R}^n} (v(x + x' - y) - v(x - y))\psi(y) \, dy. \]

To see that this is small for \( x' \) small, we split the integral into two pieces. Since \( \psi \) is very small near infinity, given \( \epsilon > 0 \) we can choose \( R \) so large that
\[ ||v||_\infty \cdot \int_{|y| \geq R} |\psi(y)| \, dy \leq \epsilon/4. \]

The set \( |y| \leq R \) is compact and if \( |x| \leq R', |x'| \leq 1 \) then \( |x + x' - y| \leq R + R' + 1 \). A continuous function is uniformly continuous on any
compact set, so we can choose $\delta > 0$ such that

\[
(8.6) \quad \sup_{|x'| < \delta, |y| \leq R} |v(x + x' - y) - v(x - y)| \cdot \int_{|y| \leq R} |\psi(y)| \, dy < \epsilon/2.
\]

Combining (8.5) and (8.6) we conclude that $v * \psi$ is continuous. Finally, we conclude that

\[
(8.7) \quad v \in C^0_0(\mathbb{R}^n) \Rightarrow v * \psi \in C^0_0(\mathbb{R}^n).
\]

For this we need to show that $v * \psi$ is small at infinity, which follows from the fact that $v$ is small at infinity. Namely given $\epsilon > 0$ there exists $R > 0$ such that $|v(y)| \leq \epsilon$ if $|y| \geq R$. Divide the integral defining the convolution into two

\[
|v * \psi(x)| \leq \int_{|y| > R} u(y)\psi(x - y)dy + \int_{|y| < R} |u(y)\psi(x - y)|dy 
\leq \epsilon/2||\psi||_{\infty} + ||u||_{\infty} \sup_{B(x,R)} |\psi|.
\]

Since $\psi \in \mathcal{S}(\mathbb{R}^n)$ the last constant tends to 0 as $|x| \to \infty$.

We can do much better than this! Assuming $|x'| \leq 1$ we can use Taylor’s formula with remainder to write

\[
(8.8) \quad \psi(z + x') - \psi(z) = \int_0^1 \frac{d}{dt} \psi(z + tx') \, dt = \sum_{j=1}^n x_j \cdot \tilde{\psi}_j(z, x').
\]

As Problem 23 I ask you to check carefully that

\[
(8.9) \quad \psi_j(z; x') \in \mathcal{S}(\mathbb{R}^n) \text{ depends continuously on } x' \text{ in } |x'| \leq 1.
\]

Going back to (8.3)) we can use the translation and reflection-invariance of Lebesgue measure to rewrite the integral (by changing variable) as

\[
(8.10) \quad v * \psi(x) = \int_{\mathbb{R}^n} v(y)\psi(x - y) \, dy.
\]

This reverses the role of $v$ and $\psi$ and shows that if both $v$ and $\psi$ are in $\mathcal{S}(\mathbb{R}^n)$ then $v * \psi = \psi * v$.

Using this formula on (8.4) we find

\[
(8.11) \quad v * \psi(x + x') - v * \psi(x) = \int v(y)(\psi(x + x' - y) - \psi(x - y)) \, dy 
= \sum_{j=1}^n x_j \int_{\mathbb{R}^n} v(y)\tilde{\psi}_j(x - y, x') \, dy = \sum_{j=1}^n x_j (v * \psi_j(\cdot; x')(x).
\]
From (8.9) and what we have already shown, \( v * \psi(\cdot; x') \) is continuous in both variables, and is in \( C^0_0(\mathbb{R}^n) \) in the first. Thus
\[
(8.12) \quad v \in C^0_0(\mathbb{R}^n), \ \psi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow v * \psi \in C^1_0(\mathbb{R}^n).
\]
In fact we also see that
\[
(8.13) \quad \frac{\partial}{\partial x_j} v * \psi = v * \frac{\partial \psi}{\partial x_j}.
\]
Thus \( v * \psi \) inherits its regularity from \( \psi \).

**Proposition 8.1.** If \( v \in C^0_0(\mathbb{R}^n) \) and \( \psi \in \mathcal{S}(\mathbb{R}^n) \) then
\[
(8.14) \quad v * \psi \in C^\infty_0(\mathbb{R}^n) = \bigcap_{k \geq 0} C^k_0(\mathbb{R}^n).
\]
**Proof.** This follows from (8.12), (8.13) and induction. \( \square \)

Now, let us make a more special choice of \( \psi \). We have shown the existence of
\[
(8.15) \quad \varphi \in C^\infty_c(\mathbb{R}^n), \ \varphi \geq 0, \ \text{supp}(\varphi) \subset \{|x| \leq 1\}.
\]
We can also assume \( \int_{\mathbb{R}^n} \varphi \, dx = 1 \), by multiplying by a positive constant. Now consider
\[
(8.16) \quad \varphi_t(x) = t^{-n} \varphi \left( \frac{x}{t} \right), \quad 1 \geq t > 0.
\]
This has all the same properties, except that
\[
(8.17) \quad \text{supp } \varphi_t \subset \{|x| \leq t\}, \quad \int \varphi_t \, dx = 1.
\]

**Proposition 8.2.** If \( v \in C^0_0(\mathbb{R}^n) \) then as \( t \to 0 \), \( v_t = v * \varphi_t \to v \) in \( C^0_0(\mathbb{R}^n) \).

**Proof.** Using (8.17) we can write the difference as
\[
(8.18) \quad |v_t(x) - v(x)| = \left| \int_{\mathbb{R}^n} (v(x - y) - v(x)) \varphi_t(y) \, dy \right| \leq \sup_{|y| \leq t} \left| v(x - y) - v(x) \right| \to 0.
\]
Here we have used the fact that \( \varphi_t \geq 0 \) has support in \( |y| \leq t \) and has integral 1. Thus \( v_t \to v \) uniformly on any set on which \( v \) is uniformly continuous, namely \( \mathbb{R}^n \! \). \( \square \)

**Corollary 8.3.** \( C^k_0(\mathbb{R}^n) \) is dense in \( C^0_0(\mathbb{R}^n) \) for any \( k \geq p \).

**Proposition 8.4.** \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( C^k_0(\mathbb{R}^n) \) for any \( k \geq 0 \).
Proof. Take \( k = 0 \) first. The subspace \( \mathcal{C}_0^0(\mathbb{R}^n) \) is dense in \( \mathcal{C}_c^0(\mathbb{R}^n) \), by cutting off outside a large ball. If \( v \in \mathcal{C}_c^0(\mathbb{R}^n) \) has support in \( \{|x| \leq R\} \) then

\[ v \ast \varphi_t \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \]

has support in \( \{|x| \leq R + 1\} \). Since \( v \ast \varphi_t \to v \) the result follows for \( k = 0 \).

For \( k \geq 1 \) the same argument works, since \( D^\alpha(v \ast \varphi_t) = (D^\alpha V) \ast \varphi_t \).

Corollary 8.5. The map from finite Radon measures

\[ (8.19) \quad M_{\text{fin}}(\mathbb{R}^n) \ni \mu \longmapsto u_\mu \in \mathcal{S}'(\mathbb{R}^n) \]

is injective.

Now, we want the same result for \( L^2(\mathbb{R}^n) \) (and maybe for \( L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \)). I leave the measure-theoretic part of the argument to you.

Proposition 8.6. Elements of \( L^2(\mathbb{R}^n) \) are “continuous in the mean” i.e.,

\[ (8.20) \quad \lim_{|t| \to 0} \int_{\mathbb{R}^n} |u(x + t) - u(x)|^2 \, dx = 0. \]

This is Problem 24.

Using this we conclude that

\[ (8.21) \quad \mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \]

is dense as before. First observe that the space of \( L^2 \) functions of compact support is dense in \( L^2(\mathbb{R}^n) \), since

\[ \lim_{R \to \infty} \int_{|x| \geq R} |u(x)|^2 \, dx = 0 \quad \forall \ u \in L^2(\mathbb{R}^n). \]

Then look back at the discussion of \( v \ast \varphi \), now \( v \) is replaced by \( u \in L^2_c(\mathbb{R}^n) \). The compactness of the support means that \( u \in L^1(\mathbb{R}^n) \) so in

\[ (8.22) \quad u \ast \varphi(x) = \int_{\mathbb{R}^n} u(x - y) \varphi(y) dy \]

the integral is absolutely convergent. Moreover

\[ |u \ast \varphi(x + x') - u \ast \varphi(x)| \]

\[ = \left| \int_{\mathbb{R}^n} u(y) (\varphi(x + x' - y) - \varphi(x - y)) \, dy \right| \]

\[ \leq C\|u\| \sup_{|y| \leq R} |\varphi(x + x' - y) - \varphi(x - y)| \to 0 \]

as \( |x| \to \infty \).
when \(|x| \leq R\) large enough. Thus \(u \ast \varphi\) is continuous and the same argument as before shows that
\[ u \ast \varphi_t \in \mathcal{S}(\mathbb{R}^n). \]

Now to see that \(u \ast \varphi_t \to u\), assuming \(u\) has compact support (or not) we estimate the integral
\[
|u \ast \varphi_t(x) - u(x)| = \left| \int (u(x-y) - u(x))\varphi_t(y) \, dy \right|
\leq \int |u(x-y) - u(x)| \varphi_t(y) \, dy.
\]

Using the same argument twice
\[
\int |u \ast \varphi_t(x) - u(x)|^2 \, dx
\leq \iint |u(x-y) - u(x)| \varphi_t(y) \, dx \, dy \, dy'
\leq \left( \int |u(x-y) - u(x)|^2 \varphi_t(y) \, dx \, dy \, dy' \right)
\leq \sup_{|y| \leq t} \int |u(x-y) - u(x)|^2 \, dx.
\]

Note that at the second step here I have used Schwarz’s inequality with the integrand written as the product
\[
|u(x-y) - u(x)| \varphi_t^{1/2}(y)\varphi_t^{1/2}(y') \cdot |u(x-y') - u(x)| \varphi_t^{1/2}(y)\varphi_t^{1/2}(y').
\]
Thus we now know that
\[ L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \] is injective.

This means that all our usual spaces of functions ‘sit inside’ \(\mathcal{S}'(\mathbb{R}^n)\).

Finally we can use convolution with \(\varphi_t\) to show the existence of smooth partitions of unity. If \(K \subseteq U \subseteq \mathbb{R}^n\) is a compact set in an open set then we have shown the existence of \(\xi \in \mathcal{C}_c^0(\mathbb{R}^n)\), with \(\xi = 1\) in some neighborhood of \(K\) and \(\xi = 1\) in some neighborhood of \(K\) and \(\text{supp}(\xi) \subseteq U\).

Then consider \(\xi \ast \varphi_t\) for \(t\) small. In fact
\[
\text{supp}(\xi \ast \varphi_t) \subseteq \{ p \in \mathbb{R}^n : \text{dist}(p, \text{supp} \xi) \leq 2t \}
\] and similarly, \(0 \leq \xi \ast \varphi_t \leq 1\) and
\[
\xi \ast \varphi_t = 1 \text{ at } p \text{ if } \xi = 1 \text{ on } B(p, 2t).
\]
Using this we get:
Proposition 8.7. If $U_a \subset \mathbb{R}^n$ are open for $a \in A$ and $K \Subset \bigcup_{a \in A} U_a$ then there exist finitely many $\varphi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, with $0 \leq \varphi_i \leq 1$, $\text{supp}(\varphi_i) \subset U_a$, such that $\sum_i \varphi_i = 1$ in a neighborhood of $K$.

Proof. By the compactness of $K$ we may choose a finite open subcover. Using Lemma 1.8 we may choose a continuous partition, $\phi'_i$, of unity subordinate to this cover. Using the convolution argument above we can replace $\phi'_i$ by $\phi'_i \ast \varphi_t$ for $t > 0$. If $t$ is sufficiently small then this is again a partition of unity subordinate to the cover, but now smooth.

Next we can make a simple 'cut off argument' to show

Lemma 8.8. The space $\mathcal{C}_c^\infty(\mathbb{R}^n)$ of $\mathcal{C}^\infty$ functions of compact support is dense in $\mathcal{S}(\mathbb{R}^n)$.

Proof. Choose $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ in $|x| \leq 1$. Then given $\psi \in \mathcal{S}(\mathbb{R}^n)$ consider the sequence $\psi_n(x) = \varphi(x/n) \psi(x)$. Clearly $\psi_n = \psi$ on $|x| \leq n$, so if it converges in $\mathcal{S}(\mathbb{R}^n)$ it must converge to $\psi$. Suppose $m \geq n$ then by Leibniz’s formula

$$D_x^\alpha (\psi_n(x) - \psi_m(x)) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D_x^\beta \left( \frac{x}{n} \right) \cdot \psi(x).$$

All derivatives of $\varphi(x/n)$ are bounded, independent of $n$ and $\psi_n = \psi_m$ in $|x| \leq n$ so for any $p$

$$|D_x^\alpha (\psi_n(x) - \psi_m(x))| \leq \begin{cases} 0 & |x| \leq n \\ C_{\alpha,p} |x|^{-2p} & |x| \geq n \end{cases}.$$ 

Hence $\psi_n$ is Cauchy in $\mathcal{S}(\mathbb{R}^n)$.

Thus every element of $\mathcal{S}'(\mathbb{R}^n)$ is determined by its restriction to $\mathcal{C}_c^\infty(\mathbb{R}^n)$. The support of a tempered distribution was defined above to be

$$\text{supp}(u) = \{ x \in \mathbb{R}^n; \ \exists \ varphi \in \mathcal{S}(\mathbb{R}^n), \ \varphi(x) \neq 0, \ \varphi u = 0 \}^c.$$

Using the preceding lemma and the construction of smooth partitions of unity we find

Proposition 8.9. If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\text{supp}(u) = \emptyset$ then $u = 0$. 

\[\text{Problem 25.}\]
Proof. From (8.23), if \( \psi \in \mathcal{S}(\mathbb{R}^n) \), \( \text{supp}(\psi u) \subset \text{supp}(u) \). If \( x \ni \text{supp}(u) \) then, by definition, \( \varphi u = 0 \) for some \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) with \( \varphi(x) \neq 0 \). Thus \( \varphi \neq 0 \) on \( B(x, \epsilon) \) for \( \epsilon > 0 \) sufficiently small. If \( \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) has support in \( B(x, \epsilon) \) then \( \psi u = \psi \varphi u = 0 \), where \( \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \):

\[
\tilde{\psi} = \begin{cases} 
\psi/\varphi & \text{in } B(x, \epsilon) \\
0 & \text{elsewhere}.
\end{cases}
\]

Thus, given \( K \in \mathbb{R}^n \) we can find \( \varphi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n) \), supported in such balls, so that \( \sum_j \varphi_j \equiv 1 \) on \( K \) but \( \varphi_j u = 0 \). For given \( \mu \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) apply this to \( \text{supp}(\mu) \). Then

\[
\mu = \sum_j \varphi_j \mu \Rightarrow u(\mu) = \sum_j (\phi_j u)(\mu) = 0.
\]

Thus \( u = 0 \) on \( \mathcal{C}_c^\infty(\mathbb{R}^n) \), so \( u = 0 \).

The linear space of distributions of compact support will be denoted \( \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \); it is often written \( \mathcal{E}'(\mathbb{R}^n) \).

Now let us give a characterization of the ‘delta function’

\[
\delta(\varphi) = \varphi(0) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n),
\]

or at least the one-dimensional subspace of \( \mathcal{S}'(\mathbb{R}^n) \) it spans. This is based on the simple observation that \( (x_j \varphi)(0) = 0 \) if \( \varphi \in \mathcal{S}(\mathbb{R}^n) \)!

**Proposition 8.10.** If \( u \in \mathcal{S}'(\mathbb{R}^n) \) satisfies \( x_j u = 0 \), \( j = 1, \cdots, n \) then \( u = c\delta \).

**Proof.** The main work is in characterizing the null space of \( \delta \) as a linear functional, namely in showing that

(8.24) \( \mathcal{H} = \{ \varphi \in \mathcal{S}(\mathbb{R}^n); \ \varphi(0) = 0 \} \)

can also be written as

(8.25) \( \mathcal{H} = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n); \ \varphi = \sum_{j=1}^{n} x_j \psi_j, \ \varphi_j \in \mathcal{S}(\mathbb{R}^n) \right\} \).

Clearly the right side of (8.25) is contained in the left. To see the converse, suppose first that

(8.26) \( \varphi \in \mathcal{S}(\mathbb{R}^n), \ \varphi = 0 \text{ in } |x| < 1. \)

Then define

\[
\psi = \begin{cases} 
0 & |x| < 1 \\
\varphi/|x|^2 & |x| \geq 1.
\end{cases}
\]
All the derivatives of $1/|x|^2$ are bounded in $|x| \geq 1$, so from Leibniz’s formula it follows that $\psi \in \mathcal{S}(\mathbb{R}^n)$. Since
\[ \varphi = \sum_j x_j(x_j \psi) \]
this shows that $\varphi$ of the form (8.26) is in the right side of (8.25). In general suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then
\[ \varphi(x) - \varphi(0) = \int_0^t \frac{d}{dt} \varphi(tx) \, dt \]
(8.27)
\[ = \sum_j x_j \int_0^t \frac{\partial \varphi}{\partial x_j}(tx) \, dt. \]
Certainly these integrals are $C^\infty$, but they may not decay rapidly at infinity. However, choose $\mu \in C^\infty_0(\mathbb{R}^n)$ with $\mu = 1$ in $|x| \leq 1$. Then (8.27) becomes, if $\varphi(0) = 0$,
\[ \varphi = \mu \varphi + (1 - \mu) \varphi \\
= \sum_{j=1}^n x_j \psi_j + (1 - \mu) \varphi, \quad \psi_j = \mu \int_0^t \frac{\partial \varphi}{\partial x_j}(tx) \, dt \in \mathcal{S}(\mathbb{R}^n). \]
Since $(1 - \mu) \varphi$ is of the form (8.26), this proves (8.25).

Our assumption on $u$ is that $x_j u = 0$, thus
\[ u(\varphi) = 0 \quad \forall \ \varphi \in \mathcal{H} \]
by (8.25). Choosing $\mu$ as above, a general $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written
\[ \varphi = \varphi(0) \cdot \mu + \varphi', \ \varphi' \in \mathcal{H}. \]
Then
\[ u(\varphi) = \varphi(0) u(\mu) \Rightarrow u = c \delta, \ c = u(\mu). \]

This result is quite powerful, as we shall soon see. The Fourier transform of an element $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is\(^{14}\)
\[ \hat{\varphi}(\xi) = \int e^{-ix \cdot \xi} \varphi(x) \, dx, \ \xi \in \mathbb{R}^n. \]

\(^{14}\)Normalizations vary, but it doesn’t matter much.
The integral certainly converges, since $|\varphi| \leq C(x)^{-n-1}$. In fact it follows easily that $\hat{\varphi}$ is continuous, since
\[
|\hat{\varphi}(\xi) - \hat{\varphi}(\xi')| \leq \int \left| e^{ix-\xi} - e^{-x}\xi' \right| |\varphi| \, dx \\
\to 0 \text{ as } \xi' \to \xi .
\]
In fact

**Proposition 8.11.** Fourier transformation, (8.28), defines a continuous linear map
\[
(F : S(\mathbb{R}^n) \to S(\mathbb{R}^n), \mathcal{F}\varphi = \hat{\varphi} .
\]

**Proof.** Differentiating under the integral\(^{15}\) sign shows that
\[
\partial_{\xi_j} \hat{\varphi}(\xi) = -i \int e^{-ix-\xi} x_j \varphi(x) \, dx .
\]
Since the integral on the right is absolutely convergent that shows that (remember the $i$'s)
\[
(D_{\xi_j} \hat{\varphi} = -\hat{x_j} \varphi, \ \forall \varphi \in S(\mathbb{R}^n) .
\]
Similarly, if we multiply by $\xi_j$ and observe that $\xi_j e^{-ix-\xi} = i \frac{\partial}{\partial x_j} e^{-ix-\xi}$ then integration by parts shows
\[
\xi_j \hat{\varphi} = i \int \left( \frac{\partial}{\partial x_j} e^{-ix-\xi} \right) \varphi(x) \, dx \\
= -i \int e^{-ix-\xi} \frac{\partial \varphi}{\partial x_j} \, dx \\
= \hat{D_j} \varphi = \xi_j \hat{\varphi}, \ \forall \varphi \in S(\mathbb{R}^n) .
\]

Since $x_j \varphi, D_j \varphi \in S(\mathbb{R}^n)$ these results can be iterated, showing that
\[
(8.32) \quad \xi^\alpha D^\beta \hat{\varphi} = \mathcal{F} \left( (-1)^{|\beta|} D^\alpha x^\beta \varphi \right) .
\]
Thus $\left| \xi^\alpha D^\beta \hat{\varphi} \right| \leq C_{\alpha \beta} \sup \left| (x)^{n+1} D^\alpha x^\beta \varphi \right| \leq C \left( (x)^{n+1+|\beta|} \varphi \right)_{C^{n+1}}$, which shows that $\mathcal{F}$ is continuous as a map (8.32).

Suppose $\varphi \in S(\mathbb{R}^n)$. Since $\hat{\varphi} \in S(\mathbb{R}^n)$ we can consider the distribution $u \in S'(\mathbb{R}^n)$
\[
(8.33) \quad u(\varphi) = \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \, d\xi .
\]
\(^{15}\)See [5]
The continuity of \( u \) follows from the fact that integration is continuous and (8.29). Now observe that

\[
    u(x_j \varphi) = \int \mathbb{R}^n \overline{x_j \varphi}(\xi) \, d\xi
    = - \int \mathbb{R}^n D_\xi \hat{\varphi} \, d\xi = 0
\]

where we use (8.30). Applying Proposition 8.10 we conclude that \( u = c \delta \) for some (universal) constant \( c \). By definition this means

\[
    \int \varphi(\xi) \, d\xi = c \varphi(0). \tag{8.34}
\]

So what is the constant? To find it we need to work out an example. The simplest one is

\[
    \varphi = \exp(-|x|^2/2). \tag{8.35}
\]

Lemma 8.12. The Fourier transform of the Gaussian \( \exp(-|x|^2/2) \) is the Gaussian \( (2\pi)^{n/2} \exp(-|\xi|^2/2) \).

Proof. There are two obvious methods — one uses complex analysis (Cauchy’s theorem) the other, which I shall follow, uses the uniqueness of solutions to ordinary differential equations.

First observe that \( \exp(-|x|^2/2) = \prod_j \exp(-x_j^2/2) \). Thus\(^\text{16}\)

\[
    \hat{\varphi}(\xi) = \prod_{j=1}^n \hat{\psi}(\xi_j), \quad \psi(x) = e^{-x^2/2},
\]

being a function of one variable. Now \( \psi \) satisfies the differential equation

\[
    (\partial_x + x) \psi = 0,
\]

and is the only solution of this equation up to a constant multiple. By (8.30) and (8.31) its Fourier transform satisfies

\[
    \widehat{\partial_x \psi + x \psi} = i\xi \hat{\psi} + i \frac{d}{d\xi} \hat{\psi} = 0.
\]

This is the same equation, but in the \( \xi \) variable. Thus \( \hat{\psi} = ce^{-|\xi|^2/2} \). Again we need to find the constant. However,

\[
    \hat{\psi}(0) = c = \int e^{-x^2/2} \, dx = (2\pi)^{1/2}
\]

\(\text{16}\)Really by Fubini’s theorem, but here one can use Riemann integrals.
by the standard use of polar coordinates:

\[ c^2 = \int_{\mathbb{R}^n} e^{-(x^2+y^2)/2} \, dx \, dy = \int_0^\infty \int_0^{2\pi} e^{-r^2/2} r \, dr \, d\theta = 2\pi. \]

This proves the lemma.

Thus we have shown that for any \( \varphi \in S(\mathbb{R}^n) \)

(8.35) \[ \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \, d\xi = (2\pi)^n \varphi(0). \]

Since this is true for \( \varphi = \exp(-|x|^2/2) \). The identity allows us to invert the Fourier transform.