

7. TEMPERED DISTRIBUTIONS

A good first reference for distributions is [2], [4] gives a more exhaustive treatment.

The complete metric topology on $\mathcal{S}(\mathbb{R}^n)$ is described above. Next I want to try to convince you that elements of its dual space $\mathcal{S}'(\mathbb{R}^n)$, have enough of the properties of functions that we can work with them as ‘generalized functions’.

First let me develop some notation. A differentiable function $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ has partial derivatives which we have denoted $\partial\varphi/\partial x_j : \mathbb{R}^n \rightarrow \mathbb{C}$. For reasons that will become clear later, we put a $\sqrt{-1}$ into the definition and write

$$(7.1) \quad D_j\varphi = \frac{1}{i} \frac{\partial\varphi}{\partial x_j}.$$

We say φ is once continuously differentiable if each of these $D_j\varphi$ is continuous. Then we defined k times continuous differentiability inductively by saying that φ and the $D_j\varphi$ are $(k-1)$ -times continuously differentiable. For $k=2$ this means that

$$D_j D_k \varphi \text{ are continuous for } j, k = 1, \dots, n.$$

Now, recall that, if continuous, these second derivatives are symmetric:

$$(7.2) \quad D_j D_k \varphi = D_k D_j \varphi.$$

This means we can use a compact notation for higher derivatives. Put $\mathbb{N}_0 = \{0, 1, \dots\}$; we call an element $\alpha \in \mathbb{N}_0^n$ a ‘multi-index’ and if φ is at least k times continuously differentiable, we set¹²

$$(7.3) \quad D^\alpha \varphi = \frac{1}{i^{|\alpha|}} \frac{\partial^{\alpha_1}}{\partial x_1} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \varphi \text{ whenever } |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k.$$

Now we have *defined* the spaces.

$$(7.4) \quad \mathcal{C}_0^k(\mathbb{R}^n) = \{ \varphi : \mathbb{R}^n \rightarrow \mathbb{C}; D^\alpha \varphi \in \mathcal{C}_0^0(\mathbb{R}^n) \forall |\alpha| \leq k \}.$$

Notice the convention is that $D^\alpha \varphi$ is asserted to exist if it is required to be continuous! Using $\langle x \rangle = (1 + |x|^2)^{1/2}$ we defined

$$(7.5) \quad \langle x \rangle^{-k} \mathcal{C}_0^k(\mathbb{R}^n) = \{ \varphi : \mathbb{R}^n \rightarrow \mathbb{C}; \langle x \rangle^k \varphi \in \mathcal{C}_0^k(\mathbb{R}^n) \},$$

and then our space of test functions is

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x \rangle^{-k} \mathcal{C}_0^k(\mathbb{R}^n).$$

¹²Periodically there is the possibility of confusion between the two meanings of $|\alpha|$ but it seldom arises.

Thus,

$$(7.6) \quad \varphi \in \mathcal{S}(\mathbb{R}^n) \Leftrightarrow D^\alpha(\langle x \rangle^k \varphi) \in \mathcal{C}_0^0(\mathbb{R}^n) \quad \forall |\alpha| \leq k \text{ and all } k.$$

Lemma 7.1. *The condition $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written*

$$\langle x \rangle^k D^\alpha \varphi \in \mathcal{C}_0^0(\mathbb{R}^n) \quad \forall |\alpha| \leq k, \forall k.$$

Proof. We first check that

$$\begin{aligned} \varphi \in \mathcal{C}_0^0(\mathbb{R}^n), \quad D_j(\langle x \rangle \varphi) \in \mathcal{C}_0^0(\mathbb{R}^n), \quad j = 1, \dots, n \\ \Leftrightarrow \varphi \in \mathcal{C}_0^0(\mathbb{R}^n), \quad \langle x \rangle D_j \varphi \in \mathcal{C}_0^0(\mathbb{R}^n), \quad j = 1, \dots, n. \end{aligned}$$

Since

$$D_j \langle x \rangle \varphi = \langle x \rangle D_j \varphi + (D_j \langle x \rangle) \varphi$$

and $D_j \langle x \rangle = \frac{1}{i} x_j \langle x \rangle^{-1}$ is a bounded continuous function, this is clear.

Then consider the same thing for a larger k :

$$(7.7) \quad \begin{aligned} D^\alpha \langle x \rangle^p \varphi \in \mathcal{C}_0^0(\mathbb{R}^n) \quad \forall |\alpha| = p, \quad 0 \leq p \leq k \\ \Leftrightarrow \langle x \rangle^p D^\alpha \varphi \in \mathcal{C}_0^0(\mathbb{R}^n) \quad \forall |\alpha| = p, \quad 0 \leq p \leq k. \end{aligned}$$

□

I leave you to check this as Problem 7.1.

Corollary 7.2. *For any $k \in \mathbb{N}$ the norms*

$$\|\langle x \rangle^k \varphi\|_{\mathcal{C}^k} \text{ and } \sum_{\substack{|\alpha| \leq k, \\ |\beta| \leq k}} \|x^\alpha D_x^\beta \varphi\|_\infty$$

are equivalent.

Proof. Any reasonable proof of (7.2) shows that the norms

$$\|\langle x \rangle^k \varphi\|_{\mathcal{C}^k} \text{ and } \sum_{|\beta| \leq k} \|\langle x \rangle^k D^\beta \varphi\|_\infty$$

are equivalent. Since there are positive constants such that

$$C_1 \left(1 + \sum_{|\alpha| \leq k} |x^\alpha| \right) \leq \langle x \rangle^k \leq C_2 \left(1 + \sum_{|\alpha| \leq k} |x^\alpha| \right)$$

the equivalent of the norms follows.

□

Proposition 7.3. *A linear functional $u : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$ is continuous if and only if there exist C, k such that*

$$|u(\varphi)| \leq C \sum_{\substack{|\alpha| \leq k, \\ |\beta| \leq k}} \sup_{\mathbb{R}^n} |x^\alpha D_x^\beta \varphi|.$$

Proof. This is just the equivalence of the norms, since we showed that $u \in \mathcal{S}'(\mathbb{R}^n)$ if and only if

$$|u(\varphi)| \leq C \|\langle x \rangle^k \varphi\|_{C^k}$$

for some k . □

Lemma 7.4. *A linear map*

$$T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$$

is continuous if and only if for each k there exist C and j such that if $|\alpha| \leq k$ and $|\beta| \leq j$

$$(7.8) \quad \sup |x^\alpha D^\beta T\varphi| \leq C \sum_{|\alpha'| \leq j, |\beta'| \leq j} \sup_{\mathbb{R}^n} |x^{\alpha'} D^{\beta'} \varphi| \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Proof. This is Problem 7.2. □

All this messing about with norms shows that

$$x_j : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n) \text{ and } D_j : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$$

are continuous.

So now we have some idea of what $u \in \mathcal{S}'(\mathbb{R}^n)$ means. Let's notice that $u \in \mathcal{S}'(\mathbb{R}^n)$ implies

$$(7.9) \quad x_j u \in \mathcal{S}'(\mathbb{R}^n) \quad \forall j = 1, \dots, n$$

$$(7.10) \quad D_j u \in \mathcal{S}'(\mathbb{R}^n) \quad \forall j = 1, \dots, n$$

$$(7.11) \quad \varphi u \in \mathcal{S}'(\mathbb{R}^n) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

where we have to *define* these things in a reasonable way. Remember that $u \in \mathcal{S}'(\mathbb{R}^n)$ is “supposed” to be like an integral against a “generalized function”

$$(7.12) \quad u(\psi) = \int_{\mathbb{R}^n} u(x)\psi(x) dx \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n).$$

Since it would be true if u were a function we *define*

$$(7.13) \quad x_j u(\psi) = u(x_j \psi) \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n).$$

Then we check that $x_j u \in \mathcal{S}'(\mathbb{R}^n)$:

$$\begin{aligned} |x_j u(\psi)| &= |u(x_j \psi)| \\ &\leq C \sum_{|\alpha| \leq k, |\beta| \leq k} \sup_{\mathbb{R}^n} |x^\alpha D^\beta (x_j \psi)| \\ &\leq C' \sum_{|\alpha| \leq k+1, |\beta| \leq k} \sup_{\mathbb{R}^n} |x^\alpha D^\beta \psi|. \end{aligned}$$

Similarly we can define the partial *derivatives* by using the standard integration by parts formula

$$(7.14) \quad \int_{\mathbb{R}^n} (D_j u)(x) \varphi(x) dx = - \int_{\mathbb{R}^n} u(x) (D_j \varphi(x)) dx$$

if $u \in \mathcal{C}_0^1(\mathbb{R}^n)$. Thus if $u \in \mathcal{S}'(\mathbb{R}^n)$ again we *define*

$$D_j u(\psi) = -u(D_j \psi) \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n).$$

Then it is clear that $D_j u \in \mathcal{S}'(\mathbb{R}^n)$.

Iterating these definition we find that D^α , for any multi-index α , defines a linear map

$$(7.15) \quad D^\alpha : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n).$$

In general a linear differential operator with constant coefficients is a sum of such “monomials”. For example Laplace’s operator is

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2} = D_1^2 + D_2^2 + \dots + D_n^2.$$

We will be interested in trying to solve differential equations such as

$$\Delta u = f \in \mathcal{S}'(\mathbb{R}^n).$$

We can also multiply $u \in \mathcal{S}'(\mathbb{R}^n)$ by $\varphi \in \mathcal{S}(\mathbb{R}^n)$, simply defining

$$(7.16) \quad \varphi u(\psi) = u(\varphi \psi) \quad \forall \psi \in \mathcal{S}(\mathbb{R}^n).$$

For this to make sense it suffices to check that

$$(7.17) \quad \sum_{\substack{|\alpha| \leq k, \\ |\beta| \leq k}} \sup_{\mathbb{R}^n} |x^\alpha D^\beta(\varphi \psi)| \leq C \sum_{\substack{|\alpha| \leq k, \\ |\beta| \leq k}} \sup_{\mathbb{R}^n} |x^\alpha D^\beta \psi|.$$

This follows easily from Leibniz’ formula.

Now, to start thinking of $u \in \mathcal{S}'(\mathbb{R}^n)$ as a generalized function we first define its *support*. Recall that

$$(7.18) \quad \text{supp}(\psi) = \text{clos} \{x \in \mathbb{R}^n; \psi(x) \neq 0\}.$$

We can write this in another ‘weak’ way which is easier to generalize. Namely

$$(7.19) \quad p \notin \text{supp}(u) \Leftrightarrow \exists \varphi \in \mathcal{S}(\mathbb{R}^n), \varphi(p) \neq 0, \varphi u = 0.$$

In fact this definition makes sense for *any* $u \in \mathcal{S}'(\mathbb{R}^n)$.

Lemma 7.5. *The set $\text{supp}(u)$ defined by (7.19) is a closed subset of \mathbb{R}^n and reduces to (7.18) if $u \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. The set defined by (7.19) is closed, since

$$(7.20) \quad \text{supp}(u)^{\mathfrak{G}} = \{p \in \mathbb{R}^n; \exists \varphi \in \mathcal{S}(\mathbb{R}^n), \varphi(p) \neq 0, \varphi u = 0\}$$

is clearly open — the same φ works for nearby points. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ we define $u_\psi \in \mathcal{S}'(\mathbb{R}^n)$, which we will again identify with ψ , by

$$(7.21) \quad u_\psi(\varphi) = \int \varphi(x)\psi(x) dx.$$

Obviously $u_\psi = 0 \implies \psi = 0$, simply set $\varphi = \bar{\psi}$ in (7.21). Thus the map

$$(7.22) \quad \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto u_\psi \in \mathcal{S}'(\mathbb{R}^n)$$

is injective. We want to show that

$$(7.23) \quad \text{supp}(u_\psi) = \text{supp}(\psi)$$

on the left given by (7.19) and on the right by (7.18). We show first that

$$\text{supp}(u_\psi) \subset \text{supp}(\psi).$$

Thus, we need to see that $p \notin \text{supp}(\psi) \implies p \notin \text{supp}(u_\psi)$. The first condition is that $\psi(x) = 0$ in a neighbourhood, U of p , hence there is a \mathcal{C}^∞ function φ with support in U and $\varphi(p) \neq 0$. Then $\varphi\psi \equiv 0$. Conversely suppose $p \notin \text{supp}(u_\psi)$. Then there exists $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(p) \neq 0$ and $\varphi u_\psi = 0$, i.e., $\varphi u_\psi(\eta) = 0 \forall \eta \in \mathcal{S}(\mathbb{R}^n)$. By the injectivity of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ this means $\varphi\psi = 0$, so $\psi \equiv 0$ in a neighborhood of p and $p \notin \text{supp}(\psi)$. \square

Consider the simplest examples of distribution which are not functions, namely those with support at a given point p . The obvious one is the Dirac delta ‘function’

$$(7.24) \quad \delta_p(\varphi) = \varphi(p) \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We can make many more, because D^α is *local*

$$(7.25) \quad \text{supp}(D^\alpha u) \subset \text{supp}(u) \forall u \in \mathcal{S}'(\mathbb{R}^n).$$

Indeed, $p \notin \text{supp}(u) \implies \exists \varphi \in \mathcal{S}(\mathbb{R}^n), \varphi u \equiv 0, \varphi(p) \neq 0$. Thus each of the distributions $D^\alpha \delta_p$ also has support contained in $\{p\}$. In fact none of them vanish, and they are all linearly independent.