2. Measures and $\sigma$-algebras

An outer measure such as $\mu^*$ is a rather crude object since, even if the $A_i$ are disjoint, there is generally strict inequality in (1.14). It turns out to be unreasonable to expect equality in (1.14), for disjoint unions, for a function defined on all subsets of $X$. We therefore restrict attention to smaller collections of subsets.

**Definition 2.1.** A collection of subsets $\mathcal{M}$ of a set $X$ is a $\sigma$-algebra if

1. $\phi, X \in \mathcal{M}$
2. $E \in \mathcal{M} \implies E^C = X \setminus E \in \mathcal{M}$
3. $\{E_i\}_{i=1}^\infty \subset \mathcal{M} \implies \bigcup_{i=1}^\infty E_i \in \mathcal{M}$.

For a general outer measure $\mu^*$ we define the notion of $\mu^*$-measurability of a set.

**Definition 2.2.** A set $E \subset X$ is $\mu^*$-measurable (for an outer measure $\mu^*$ on $X$) if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C) \quad \forall A \subset X.$$  

**Proposition 2.3.** The collection of $\mu^*$-measurable sets for any outer measure is a $\sigma$-algebra.

**Proof.** Suppose $E$ is $\mu^*$-measurable, then $E^C$ is $\mu^*$-measurable by the symmetry of (2.1).

Suppose $A$, $E$ and $F$ are any three sets. Then

$$A \cap (E \cup F) = (A \cap E \cap F) \cup (A \cap E \cap F^C) \cup (A \cap E^C \cap F)$$

$$A \cap (E \cup F)^C = A \cap E^C \cap F^C.$$

From the subadditivity of $\mu^*$

$$\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C)$$

$$\leq \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cup F^C)$$

$$\quad + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E \cap F^C).$$

Now, if $E$ and $F$ are $\mu^*$-measurable then applying the definition twice, for any $A$,

$$\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^C)$$

$$\quad + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E^C \cap F^C)$$

$$\geq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C).$$

The reverse inequality follows from the subadditivity of $\mu^*$, so $E \cup F$ is also $\mu^*$-measurable.
If \( \{E_i\}_{i=1}^\infty \) is a sequence of disjoint \( \mu^* \)-measurable sets, set \( F_n = \bigcup_{i=1}^n E_i \) and \( F = \bigcup_{i=1}^\infty E_i \). Then for any \( A \),
\[
\mu^*(A \cap F_n) = \mu^*(A \cap F_n \cap E_n) + \mu^*(A \cap F_n \cap E_n^C) \\
= \mu^*(A \cap E_n) + \mu^*(A \cap F_n-1).
\]
Iterating this shows that
\[
\mu^*(A \cap F_n) = \sum_{j=1}^n \mu^*(A \cap E_j).
\]

From the \( \mu^* \)-measurability of \( F_n \) and the subadditivity of \( \mu^* \),
\[
\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^C) \\
\geq \sum_{j=1}^n \mu^*(A \cap E_j) + \mu^*(A \cap F_n^C),
\]
Taking the limit as \( n \to \infty \) and using subadditivity,
\[
\mu^*(A) \geq \sum_{j=1}^\infty \mu^*(A \cap E_j) + \mu^*(A \cap F^C) \\
\geq \mu^*(A \cap F) + \mu^*(A \cap F^C) \geq \mu^*(A)
\]
proves that inequalities are equalities, so \( F \) is also \( \mu^* \)-measurable.

In general, for any countable union of \( \mu^* \)-measurable sets,
\[
\bigcup_{j=1}^\infty A_j = \bigcup_{j=1}^\infty \tilde{A}_j, \\
\tilde{A}_j = A_j \setminus \bigcup_{i=1}^{j-1} A_i = A_j \cap \left( \bigcup_{i=1}^{j-1} A_i \right)^C
\]
is \( \mu^* \)-measurable since the \( \tilde{A}_j \) are disjoint. \( \square \)

A measure (sometimes called a positive measure) is an extended function defined on the elements of a \( \sigma \)-algebra \( \mathcal{M} \):
\[
\mu : \mathcal{M} \to [0, \infty]
\]
such that
\[
\mu(\emptyset) = 0 \quad \text{and} \quad \mu \left( \bigcup_{i=1}^\infty A_i \right) = \sum_{i=1}^\infty \mu(A_i)
\]
if \( \{A_i\}_{i=1}^\infty \subset \mathcal{M} \) and \( A_i \cap A_j = \emptyset \) if \( i \neq j \).
The elements of $\mathcal{M}$ with measure zero, i.e., $E \in \mathcal{M}$, $\mu(E) = 0$, are supposed to be ‘ignorable’. The measure $\mu$ is said to be complete if
\[
(2.5) \quad E \subset X \text{ and } \exists F \in \mathcal{M}, \, \mu(F) = 0, \, E \subset F \Rightarrow E \in \mathcal{M}.
\]
See Problem 4.

The first part of the following important result due to Caratheodory was shown above.

**Theorem 2.4.** If $\mu^*$ is an outer measure on $X$ then the collection of $\mu^*$-measurable subsets of $X$ is a $\sigma$-algebra and $\mu^*$ restricted to $\mathcal{M}$ is a complete measure.

**Proof.** We have already shown that the collection of $\mu^*$-measurable subsets of $X$ is a $\sigma$-algebra. To see the second part, observe that taking $A = F$ in (2.2) gives
\[
\mu^*(F) = \sum_j \mu^*(E_j) \text{ if } F = \bigcup_{j=1}^{\infty} E_j
\]
and the $E_j$ are disjoint elements of $\mathcal{M}$. This is (2.4).

Similarly if $\mu^*(E) = 0$ and $F \subset E$ then $\mu^*(F) = 0$. Thus it is enough to show that for any subset $E \subset X$, $\mu^*(E) = 0$ implies $E \in \mathcal{M}$. For any $A \subset X$, using the fact that $\mu^*(A \cap E) = 0$, and the ‘increasing’ property of $\mu^*$
\[
\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^C) = \mu^*(A \cap E^C) \leq \mu^*(A)
\]
shows that these must always be equalities, so $E \in \mathcal{M}$ (i.e., is $\mu^*$-measurable). \qed

Going back to our primary concern, recall that we constructed the outer measure $\mu^*$ from $0 \leq u \in (\mathcal{C}_0(X))^\prime$ using (1.11) and (1.12). For the measure whose existence follows from Caratheodory’s theorem to be much use we need

**Proposition 2.5.** If $0 \leq u \in (\mathcal{C}_0(X))^\prime$, for $X$ a locally compact metric space, then each open subset of $X$ is $\mu^*$-measurable for the outer measure defined by (1.11) and (1.12) and $\mu$ in (1.11) is its measure.

**Proof.** Let $U \subset X$ be open. We only need to prove (2.1) for all $A \subset X$ with $\mu^*(A) < \infty$.\footnote{Why?}
Suppose first that \( A \subset X \) is open and \( \mu^*(A) < \infty \). Then \( A \cap U \) is open, so given \( \epsilon > 0 \) there exists \( f \in \mathcal{C}(X) \) supp\((f) \subseteq A \cap U \) with \( 0 \leq f \leq 1 \) and
\[
\mu^*(A \cap U) = \mu(A \cap U) \leq u(f) + \epsilon .
\]
Now, \( A \setminus \text{supp}(f) \) is also open, so we can find \( g \in \mathcal{C}(X) \), \( 0 \leq g \leq 1 \), supp\((g) \subseteq A \setminus \text{supp}(f) \) with
\[
\mu^*(A \setminus \text{supp}(f)) = \mu(A \setminus \text{supp}(f)) \leq u(g) + \epsilon .
\]
Since
\[
A \setminus \text{supp}(f) \supset A \cap U^C , \quad 0 \leq f + g \leq 1 , \quad \text{supp}(f + g) \subseteq A ,
\]
\[
\mu(A) \geq u(f + g) = u(f) + u(g)
\]
\[
> \mu^*(A \cap U) + \mu^*(A \cap U^C) - 2\epsilon
\]
\[
\geq \mu^*(A) - 2\epsilon
\]
using subadditivity of \( \mu^* \). Letting \( \epsilon \downarrow 0 \) we conclude that
\[
\mu^*(A) \leq \mu^*(A \cap U) + \mu^*(A \cap U^C) \leq \mu^*(A) = \mu(A) .
\]
This gives (2.1) when \( A \) is open.

In general, if \( E \subset X \) and \( \mu^*(E) < \infty \) then given \( \epsilon > 0 \) there exists \( A \subset X \) open with \( \mu^*(E) > \mu^*(A) - \epsilon \). Thus,
\[
\mu^*(E) \geq \mu^*(A \cap U) + \mu^*(A \cap U^C) - \epsilon
\]
\[
\geq \mu^*(E \cap U) + \mu^*(E \cap U^C) - \epsilon
\]
\[
\geq \mu^*(E) - \epsilon .
\]
This shows that (2.1) always holds, so \( U \) is \( \mu^* \)-measurable if it is open.

We have already observed that \( \mu(U) = \mu^*(U) \) if \( U \) is open.

Thus we have shown that the \( \sigma \)-algebra given by Caratheodory’s theorem contains all open sets. You showed in Problem 3 that the intersection of any collection of \( \sigma \)-algebras on a given set is a \( \sigma \)-algebra.

Since \( \mathcal{P}(X) \) is always a \( \sigma \)-algebra it follows that for any collection \( \mathcal{E} \subset \mathcal{P}(X) \) there is always a smallest \( \sigma \)-algebra containing \( \mathcal{E} \), namely
\[
\mathcal{M}_\mathcal{E} = \bigcap \{ \mathcal{M} \supset \mathcal{E} ; \mathcal{M} \text{ is a } \sigma \text{-algebra} , \mathcal{M} \subset \mathcal{P}(X) \} .
\]
The elements of the smallest \( \sigma \)-algebra containing the open sets are called ‘Borel sets’. A measure defined on the \( \sigma \)-algebra of all Borel sets is called a Borel measure. This we have shown:

**Proposition 2.6.** The measure defined by (1.11), (1.12) from \( 0 \leq u \in \langle \mathcal{C}_0(X) \rangle' \) by Caratheodory’s theorem is a Borel measure.

**Proof.** This is what Proposition 2.5 says! See how easy proofs are.  \( \square \)
We can even continue in the same vein. A Borel measure is said to be *outer regular* on \( E \subset X \) if
\[
\mu(E) = \inf \{ \mu(U) ; U \supset E , U \text{ open} \} .
\]
Thus the measure constructed in Proposition 2.5 is outer regular on all Borel sets! A Borel measure is *inner regular* on \( E \) if
\[
\mu(E) = \sup \{ \mu(K) ; K \subset E , K \text{ compact} \} .
\]
Here we need to know that compact sets are Borel measurable. This is Problem 5.

**Definition 2.7.** A Radon measure (on a metric space) is a Borel measure which is outer regular on all Borel sets, inner regular on open sets and finite on compact sets.

**Proposition 2.8.** The measure defined by (1.11), (1.12) from \( 0 \leq u \in (C_0(X))' \) using Carathéodory’s theorem is a Radon measure.

**Proof.** Suppose \( K \subset X \) is compact. Let \( \chi_K \) be the characteristic function of \( K \), \( \chi_K = 1 \) on \( K \), \( \chi_K = 0 \) on \( K^C \). Suppose \( f \in C_0(X) \), \( \text{supp}(f) \Subset X \) and \( f \geq \chi_K \). Set
\[
U_\epsilon = \{ x \in X ; f(x) > 1 - \epsilon \}
\]
where \( \epsilon > 0 \) is small. Thus \( U_\epsilon \) is open, by the continuity of \( f \) and contains \( K \). Moreover, we can choose \( g \in C(X) \), \( \text{supp}(g) \Subset U_\epsilon , 0 \leq g \leq 1 \) with \( g = 1 \) near\(^3\) \( K \). Thus, \( g \leq (1 - \epsilon)^{-1}f \) and hence
\[
\mu^*(K) \leq u(g) = (1 - \epsilon)^{-1}u(f) .
\]
Letting \( \epsilon \downarrow 0 \), and using the measurability of \( K \),
\[
\mu(K) \leq u(f)
\]
\[
\Rightarrow \mu(K) = \inf \{ u(f) ; f \in C(X) , \text{supp}(f) \Subset X , f \geq \chi_K \} .
\]
In particular this implies that \( \mu(K) < \infty \) if \( K \Subset X \), but is also proves (2.7).

Let me now review a little of what we have done. We used the positive functional \( u \) to define an outer measure \( \mu^* \), hence a measure \( \mu \) and then checked the properties of the latter.

This is a pretty nice scheme; getting ahead of myself a little, let me suggest that we try it on something else.

\(^3\)Meaning in a neighborhood of \( K \).
Let us say that $Q \subset \mathbb{R}^n$ is ‘rectangular’ if it is a product of finite intervals (open, closed or half-open)

\begin{equation}
Q = \prod_{i=1}^{n} (a_i, b_i) \text{ or } a_i \leq b_i
\end{equation}

we all agree on its standard volume:

\begin{equation}
v(Q) = \prod_{i=1}^{n} (b_i - a_i) \in [0, \infty).\]
\end{equation}

Clearly if we have two such sets, $Q_1 \subset Q_2$, then $v(Q_1) \leq v(Q_2)$. Let us try to define an outer measure on subsets of $\mathbb{R}^n$ by

\begin{equation}
v^*(A) = \inf \left\{ \sum_{i=1}^{\infty} v(Q_i) ; A \subset \bigcup_{i=1}^{\infty} Q_i, \text{ Q_i rectangular} \right\}.
\end{equation}

We want to show that (2.10) does define an outer measure. This is pretty easy; certainly $v(\emptyset) = 0$. Similarly if $\{A_i\}_{i=1}^{\infty}$ are (disjoint) sets and $\{Q_{ij}\}_{i=1}^{\infty}$ is a covering of $A_i$ by open rectangles then all the $Q_{ij}$ together cover $A = \bigcup_i A_i$ and

\[ v^*(A) \leq \sum_i \sum_j v(Q_{ij}) \]

\[ \Rightarrow v^*(A) \leq \sum_i v^*(A_i). \]

So we have an outer measure. We also want

**Lemma 2.9.** If $Q$ is rectangular then $v^*(Q) = v(Q)$.

Assuming this, the measure defined from $v^*$ using Caratheodory’s theorem is called Lebesgue measure.

**Proposition 2.10.** Lebesgue measure is a Borel measure.

To prove this we just need to show that (open) rectangular sets are $v^*$-measurable.