

16. SPECTRAL THEOREM

For a bounded operator T on a Hilbert space we define the spectrum as the set

$$(16.1) \quad \text{spec}(T) = \{z \in \mathbb{C}; T - z \text{Id is not invertible}\}.$$

Proposition 16.1. *For any bounded linear operator on a Hilbert space $\text{spec}(T) \subset \mathbb{C}$ is a compact subset of $\{|z| \leq \|T\|\}$.*

Proof. We show that the set $\mathbb{C} \setminus \text{spec}(T)$ (generally called the resolvent set of T) is open and contains the complement of a sufficiently large ball. This is based on the convergence of the Neumann series. Namely if T is bounded and $\|T\| < 1$ then

$$(16.2) \quad (\text{Id} - T)^{-1} = \sum_{j=0}^{\infty} T^j$$

converges to a bounded operator which is a two-sided inverse of $\text{Id} - T$. Indeed, $\|T^j\| \leq \|T\|^j$ so the series is convergent and composing with $\text{Id} - T$ on either side gives a telescoping series reducing to the identity.

Applying this result, we first see that

$$(16.3) \quad (T - z) = -z(\text{Id} - T/z)$$

is invertible if $|z| > \|T\|$. Similarly, if $(T - z_0)^{-1}$ exists for some $z_0 \in \mathbb{C}$ then

$$(16.4) \quad (T - z) = (T - z_0) - (z - z_0) = (T - z_0)^{-1}(\text{Id} - (z - z_0)(T - z_0)^{-1})$$

exists for $|z - z_0| \|(T - z_0)^{-1}\| < 1$. \square

In general it is rather difficult to precisely locate $\text{spec}(T)$.

However for a bounded self-adjoint operator it is easier. One sign of this is the the norm of the operator has an alternative, simple, characterization. Namely

$$(16.5) \quad \text{if } A^* = A \text{ then } \sup_{\|\phi\|=1} \langle A\phi, \phi \rangle = \|A\|.$$

If a is this supremum, then clearly $a \leq \|A\|$. To see the converse, choose any $\phi, \psi \in H$ with norm 1 and then replace ψ by $e^{i\theta}\psi$ with θ chosen so that $\langle A\phi, \psi \rangle$ is real. Then use the polarization identity to write

$$(16.6) \quad 4\langle A\phi, \psi \rangle = \langle A(\phi + \psi), (\phi + \psi) \rangle - \langle A(\phi - \psi), (\phi - \psi) \rangle \\ + i\langle A(\phi + i\psi), (\phi + i\psi) \rangle - i\langle A(\phi - i\psi), (\phi - i\psi) \rangle.$$

Now, by the assumed reality we may drop the last two terms and see that

$$(16.7) \quad 4|\langle A\phi, \psi \rangle| \leq a(\|\phi + \psi\|^2 + \|\phi - \psi\|^2) = 2a(\|\phi\|^2 + \|\psi\|^2) = 4a.$$

Thus indeed $\|A\| = \sup_{\|\phi\|=\|\psi\|=1} |\langle A\phi, \psi \rangle| = a$.

We can always subtract a real constant from A so that $A' = A - t$ satisfies

$$(16.8) \quad - \inf_{\|\phi\|=1} \langle A'\phi, \phi \rangle = \sup_{\|\phi\|=1} \langle A'\phi, \phi \rangle = \|A'\|.$$

Then, it follows that $A' \pm \|A'\|$ is not invertible. Indeed, there exists a sequence ϕ_n , with $\|\phi_n\| = 1$ such that $\langle (A' - \|A'\|)\phi_n, \phi_n \rangle \rightarrow 0$. Thus

$$(16.9) \quad \|(A' - \|A'\|)\phi_n\|^2 = -2\langle A'\phi_n, \phi_n \rangle + \|A'\phi_n\|^2 + \|A'\|^2 \leq -2\langle A'\phi_n, \phi_n \rangle + 2\|A'\|^2 \rightarrow 0.$$

This shows that $A' - \|A'\|$ cannot be invertible and the same argument works for $A' + \|A'\|$. For the original operator A if we set

$$(16.10) \quad m = \inf_{\|\phi\|=1} \langle A\phi, \phi \rangle \quad M = \sup_{\|\phi\|=1} \langle A\phi, \phi \rangle$$

then we conclude that neither $A - m \text{Id}$ nor $A - M \text{Id}$ is invertible and $\|A\| = \max(-m, M)$.

Proposition 16.2. *If A is a bounded self-adjoint operator then, with m and M defined by (16.10),*

$$(16.11) \quad \{m\} \cup \{M\} \subset \text{spec}(A) \subset [m, M].$$

Proof. We have already shown the first part, that m and M are in the spectrum so it remains to show that $A - z$ is invertible for all $z \in \mathbb{C} \setminus [m, M]$.

Using the self-adjointness

$$(16.12) \quad \text{Im}\langle (A - z)\phi, \phi \rangle = -\text{Im } z \|\phi\|^2.$$

This implies that $A - z$ is invertible if $z \in \mathbb{C} \setminus \mathbb{R}$. First it shows that $(A - z)\phi = 0$ implies $\phi = 0$, so $A - z$ is injective. Secondly, the range is closed. Indeed, if $(A - z)\phi_n \rightarrow \psi$ then applying (16.12) directly shows that $\|\phi_n\|$ is bounded and so can be replaced by a weakly convergent subsequence. Applying (16.12) again to $\phi_n - \phi_m$ shows that the sequence is actually Cauchy, hence converges to ϕ so $(A - z)\phi = \psi$ is in the range. Finally, the orthocomplement to this range is the null space of $A^* - \bar{z}$, which is also trivial, so $A - z$ is an isomorphism and (16.12) also shows that the inverse is bounded, in fact

$$(16.13) \quad \|(A - z)^{-1}\| \leq \frac{1}{|\text{Im } z|}.$$

When $z \in \mathbb{R}$ we can replace A by A' satisfying (16.8). Then we have to show that $A' - z$ is invertible for $|z| > \|A\|$, but that is shown in the proof of Proposition 16.1. \square

The basic estimate leading to the spectral theorem is:

Proposition 16.3. *If A is a bounded self-adjoint operator and p is a real polynomial in one variable,*

$$(16.14) \quad p(t) = \sum_{i=0}^N c_i t^i, \quad c_N \neq 0,$$

then $p(A) = \sum_{i=0}^N c_i A^i$ satisfies

$$(16.15) \quad \|p(A)\| \leq \sup_{t \in [m, M]} |p(t)|.$$

Proof. Clearly, $p(A)$ is a bounded self-adjoint operator. If $s \notin p([m, M])$ then $p(A) - s$ is invertible. Indeed, the roots of $p(t) - s$ must not lie in $[m, M]$, since otherwise $s \in p([m, M])$. Thus, factorizing $p(t) - s$ we have

$$(16.16) \quad p(t) - s = c_N \prod_{i=1}^N (t - t_i(s)), \quad t_i(s) \notin [m, M] \implies (p(A) - s)^{-1} \text{ exists}$$

since $p(A) - s = c_N \prod_i (A - t_i(s))$ and each of the factors is invertible.

Thus $\text{spec}(p(A)) \subset p([m, M])$, which is an interval (or a point), and from Proposition 16.3 we conclude that $\|p(A)\| \leq \sup p([m, M])$ which is (16.15). \square

Now, reinterpreting (16.15) we have a linear map

$$(16.17) \quad \mathcal{P}(\mathbb{R}) \ni p \longmapsto p(A) \in \mathcal{B}(H)$$

from the real polynomials to the bounded self-adjoint operators which is continuous with respect to the supremum norm on $[m, M]$. Since polynomials are dense in continuous functions on finite intervals, we see that (16.17) extends by continuity to a linear map

$$(16.18) \quad \mathcal{C}([m, M]) \ni f \longmapsto f(A) \in \mathcal{B}(H), \quad \|f(A)\| \leq \|f\|_{[m, M]}, \quad fg(A) = f(A)g(A)$$

where the multiplicativity follows by continuity together with the fact that it is true for polynomials.

Now, consider any two elements $\phi, \psi \in H$. Evaluating $f(A)$ on ϕ and pairing with ψ gives a linear map

$$(16.19) \quad \mathcal{C}([m, M]) \ni f \longmapsto \langle f(A)\phi, \psi \rangle \in \mathbb{C}.$$

This is a linear functional on $\mathcal{C}([m, M])$ to which we can apply the Riesz representatin theorem and conclude that it is defined by integration

against a unique Radon measure $\mu_{\phi,\psi}$:

$$(16.20) \quad \langle f(A)\phi, \psi \rangle = \int_{[m,M]} f d\mu_{\phi,\psi}.$$

The total mass $|\mu_{\phi,\psi}|$ of this measure is the norm of the functional. Since it is a Borel measure, we can take the integral on $-\infty, b]$ for any $b \in \mathbb{R}$ ad, with the uniqueness, this shows that we have a continuous sesquilinear map

$$(16.21) \quad P_b(\phi, \psi) : H \times H \ni (\phi, \psi) \mapsto \int_{[m,b]} d\mu_{\phi,\psi} \in \mathbb{R}, \quad |P_b(\phi, \psi)| \leq \|A\| \|\phi\| \|\psi\|.$$

From the Hilbert space Riesz representation theorem it follows that this sesquilinear form defines, and is determined by, a bounded linear operator

$$(16.22) \quad P_b(\phi, \psi) = \langle P_b \phi, \psi \rangle, \quad \|P_b\| \leq \|A\|.$$

In fact, from the functional calculus (the multiplicativity in (16.18)) we see that

$$(16.23) \quad P_b^* = P_b, \quad P_b^2 = P_b, \quad \|P_b\| \leq 1,$$

so P_b is a projection.

Thus the spectral theorem gives us an increasing (with b) family of commuting self-adjoint projections such that $\mu_{\phi,\psi}((-\infty, b]) = \langle P_b \phi, \psi \rangle$ determines the Radon measure for which (16.20) holds. One can go further and think of P_b itself as determining a measure

$$(16.24) \quad \mu((-\infty, b]) = P_b$$

which takes values in the projections on H and which allows the functions of A to be written as integrals in the form

$$(16.25) \quad f(A) = \int_{[m,M]} f d\mu$$

of which (16.20) becomes the ‘weak form’. To do so one needs to develop the theory of such measures and the corresponding integrals. This is not so hard but I shall not do it.