13. Homogeneous distributions

Next time I will talk about homogeneous distributions. On $\mathbb{R}$ the functions

$$x_t^s = \begin{cases} x^s & x > 0 \\ 0 & x < 0 \end{cases}$$

where $S \in \mathbb{R}$, is locally integrable (and hence a tempered distribution) precisely when $S > -1$. As a function it is homogeneous of degree s. Thus if $a > 0$ then

$$(ax)_t^s = a^s x_t^s \,.$$

Thinking of $x_t^s = \mu_s$ as a distribution we can set this as

$$\begin{aligned}\mu_s(ax)(\varphi) &= \int \mu_s(ax)\varphi(x)\,dx \\ &= \int \mu_s(x)\varphi(x/a)\frac{dx}{a} \\ &= a^s \mu_s(\varphi)\,.\end{aligned}$$

Thus if we *define* $\varphi_a(x) = \frac{1}{a}\varphi(\frac{x}{a})$, for any $a > 0$, $\varphi \in \mathcal{S}(\mathbb{R})$ we can ask whether a distribution is homogeneous:

$$\mu(\varphi_a) = a^s \mu(\varphi) \;\forall\; \varphi \in \mathcal{S}(\mathbb{R}).$$