

10. SOBOLEV EMBEDDING

The properties of Sobolev spaces are briefly discussed above. If m is a positive integer then $u \in H^m(\mathbb{R}^n)$ ‘means’ that u has up to m derivatives in $L^2(\mathbb{R}^n)$. The question naturally arises as to the sense in which these ‘weak’ derivatives correspond to old-fashioned ‘strong’ derivatives. Of course when m is not an integer it is a little harder to imagine what these ‘fractional derivatives’ are. However the main result is:

Theorem 10.1 (Sobolev embedding). *If $u \in H^m(\mathbb{R}^n)$ where $m > n/2$ then $u \in \mathcal{C}_0^0(\mathbb{R}^n)$, i.e.,*

$$(10.1) \quad H^m(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n), \quad m > n/2.$$

Proof. By definition, $u \in H^m(\mathbb{R}^n)$ means $v \in \mathcal{S}'(\mathbb{R}^n)$ and $\langle \xi \rangle^m \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. Suppose first that $u \in \mathcal{S}(\mathbb{R}^n)$. The Fourier inversion formula shows that

$$\begin{aligned} (2\pi)^n |u(x)| &= \left| \int e^{ix \cdot \xi} \hat{u}(\xi) d\xi \right| \\ &\leq \left(\int_{\mathbb{R}^n} \langle \xi \rangle^{2m} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \cdot \left(\sum_{\mathbb{R}^n} \langle \xi \rangle^{-2m} d\xi \right)^{1/2}. \end{aligned}$$

Now, if $m > n/2$ then the second integral is finite. Since the first integral is the norm on $H^m(\mathbb{R}^n)$ we see that

$$(10.2) \quad \sup_{\mathbb{R}^n} |u(x)| = \|u\|_{L^\infty} \leq (2\pi)^{-n} \|u\|_{H^m}, \quad m > n/2.$$

This is all for $u \in \mathcal{S}(\mathbb{R}^n)$, but $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n)$ is dense. The estimate (10.2) shows that if $u_j \rightarrow u$ in $H^m(\mathbb{R}^n)$, with $u_j \in \mathcal{S}(\mathbb{R}^n)$, then $u_j \rightarrow u'$ in $\mathcal{C}_0^0(\mathbb{R}^n)$. In fact $u' = u$ in $\mathcal{S}'(\mathbb{R}^n)$ since $u_j \rightarrow u$ in $L^2(\mathbb{R}^n)$ and $u_j \rightarrow u'$ in $\mathcal{C}_0^0(\mathbb{R}^n)$ both imply that $\int u_j \varphi$ converges, so

$$\int_{\mathbb{R}^n} u_j \varphi \rightarrow \int_{\mathbb{R}^n} u \varphi = \int_{\mathbb{R}^n} u' \varphi \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

□

Notice here the precise meaning of $u = u'$, $u \in H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, $u' \in \mathcal{C}_0^0(\mathbb{R}^n)$. When identifying $u \in L^2(\mathbb{R}^n)$ with the corresponding tempered distribution, the values on any set of measure zero ‘are lost’. Thus as *functions* (10.1) means that each $u \in H^m(\mathbb{R}^n)$ has a representative $u' \in \mathcal{C}_0^0(\mathbb{R}^n)$.

We can extend this to higher derivatives by noting that

Proposition 10.2. *If $u \in H^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, then $D^\alpha u \in H^{m-|\alpha|}(\mathbb{R}^n)$ and*

$$(10.3) \quad D^\alpha : H^m(\mathbb{R}^n) \rightarrow H^{m-|\alpha|}(\mathbb{R}^n)$$

is continuous.

Proof. First it is enough to show that each D_j defines a continuous linear map

$$(10.4) \quad D_j : H^m(\mathbb{R}^n) \rightarrow H^{m-1}(\mathbb{R}^n) \quad \forall j$$

since then (10.3) follows by composition.

If $m \in \mathbb{R}$ then $u \in H^m(\mathbb{R}^n)$ means $\hat{u} \in \langle \xi \rangle^{-m} L^2(\mathbb{R}^n)$. Since $\widehat{D_j u} = \xi_j \cdot \hat{u}$, and

$$|\xi_j| \langle \xi \rangle^{-m} \leq C_m \langle \xi \rangle^{-m+1} \quad \forall m$$

we conclude that $D_j u \in H^{m-1}(\mathbb{R}^n)$ and

$$\|D_j u\|_{H^{m-1}} \leq C_m \|u\|_{H^m}.$$

□

Applying this result we see

Corollary 10.3. *If $k \in \mathbb{N}_0$ and $m > \frac{n}{2} + k$ then*

$$(10.5) \quad H^m(\mathbb{R}^n) \subset C_0^k(\mathbb{R}^n).$$

Proof. If $|\alpha| \leq k$, then $D^\alpha u \in H^{m-k}(\mathbb{R}^n) \subset C_0^0(\mathbb{R}^n)$. Thus the ‘weak derivatives’ $D^\alpha u$ are continuous. Still we have to check that this means that u is itself k times continuously differentiable. In fact this again follows from the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$. The continuity in (10.3) implies that if $u_j \rightarrow u$ in $H^m(\mathbb{R}^n)$, $m > \frac{n}{2} + k$, then $u_j \rightarrow u'$ in $C_0^k(\mathbb{R}^n)$ (using its completeness). However $u = u'$ as before, so $u \in C_0^k(\mathbb{R}^n)$. □

In particular we see that

$$(10.6) \quad H^\infty(\mathbb{R}^n) = \bigcap_m H^m(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n).$$

These functions are not in general Schwartz test functions.

Proposition 10.4. *Schwartz space can be written in terms of weighted Sobolev spaces*

$$(10.7) \quad \mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x \rangle^{-k} H^k(\mathbb{R}^n).$$

Proof. This follows directly from (10.5) since the left side is contained in

$$\bigcap_k \langle x \rangle^{-k} \mathcal{C}_0^{k-n}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n).$$

□

Theorem 10.5 (Schwartz representation). *Any tempered distribution can be written in the form of a finite sum*

$$(10.8) \quad u = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} x^\alpha D_x^\beta u_{\alpha\beta}, \quad u_{\alpha\beta} \in \mathcal{C}_0^0(\mathbb{R}^n).$$

or in the form

$$(10.9) \quad u = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} D_x^\beta (x^\alpha v_{\alpha\beta}), \quad v_{\alpha\beta} \in \mathcal{C}_0^0(\mathbb{R}^n).$$

Thus every tempered distribution is a finite sum of derivatives of continuous functions of polynomial growth.

Proof. Essentially by definition any $u \in \mathcal{S}'(\mathbb{R}^n)$ is continuous with respect to *one* of the norms $\|\langle x \rangle^k \varphi\|_{\mathcal{C}^k}$. From the Sobolev embedding theorem we deduce that, with $m > k + n/2$,

$$|u(\varphi)| \leq C \|\langle x \rangle^k \varphi\|_{H^m} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

This is the same as

$$|\langle x \rangle^{-k} u(\varphi)| \leq C \|\varphi\|_{H^m} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

which shows that $\langle x \rangle^{-k} u \in H^{-m}(\mathbb{R}^n)$, i.e., from Proposition 9.8,

$$\langle x \rangle^{-k} u = \sum_{|\alpha| \leq m} D^\alpha u_\alpha, \quad u_\alpha \in L^2(\mathbb{R}^n).$$

In fact, choose $j > n/2$ and consider $v_\alpha \in H^j(\mathbb{R}^n)$ defined by $\hat{v}_\alpha = \langle \xi \rangle^{-j} \hat{u}_\alpha$. As in the proof of Proposition 9.14 we conclude that

$$u_\alpha = \sum_{|\beta| \leq j} D^\beta u'_{\alpha,\beta}, \quad u'_{\alpha,\beta} \in H^j(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n).$$

Thus,¹⁷

$$(10.10) \quad u = \langle x \rangle^k \sum_{|\gamma| \leq M} D_\alpha^\gamma v_\gamma, \quad v_\gamma \in \mathcal{C}_0^0(\mathbb{R}^n).$$

To get (10.9) we ‘commute’ the factor $\langle x \rangle^k$ to the inside; since I have not done such an argument carefully so far, let me do it as a lemma.

¹⁷This is probably the most useful form of the representation theorem!

Lemma 10.6. *For any $\gamma \in \mathbb{N}_0^n$ there are polynomials $p_{\alpha,\gamma}(x)$ of degrees at most $|\gamma - \alpha|$ such that*

$$\langle x \rangle^k D^\gamma v = \sum_{\alpha \leq \gamma} D^{\gamma-\alpha} (p_{\alpha,\gamma} \langle x \rangle^{k-2|\gamma-\alpha|} v).$$

Proof. In fact it is convenient to prove a more general result. Suppose p is a polynomial of a degree at most j then there exist polynomials of degrees at most $j + |\gamma - \alpha|$ such that

$$(10.11) \quad p \langle x \rangle^k D^\gamma v = \sum_{\alpha \leq \gamma} D^{\gamma-\alpha} (p_{\alpha,\gamma} \langle x \rangle^{k-2|\gamma-\alpha|} v).$$

The lemma follows from this by taking $p = 1$.

Furthermore, the identity (10.11) is trivial when $\gamma = 0$, and proceeding by induction we can suppose it is known whenever $|\gamma| \leq L$. Taking $|\gamma| = L + 1$,

$$D^\gamma = D_j D^{\gamma'} \quad |\gamma'| = L.$$

Writing the identity for γ' as

$$p \langle x \rangle^k D^{\gamma'} = \sum_{\alpha' \leq \gamma'} D^{\gamma'-\alpha'} (p_{\alpha',\gamma'} \langle x \rangle^{k-2|\gamma'-\alpha'|} v)$$

we may differentiate with respect to x_j . This gives

$$\begin{aligned} p \langle x \rangle^k D^\gamma &= -D_j (p \langle x \rangle^k) \cdot D^{\gamma'} v \\ &+ \sum_{|\alpha'| \leq \gamma'} D^{\gamma'-\alpha'} (p'_{\alpha',\gamma'} \langle x \rangle^{k-2|\gamma'-\alpha'|+2} v). \end{aligned}$$

The first term on the right expands to

$$(-(D_j p) \cdot \langle x \rangle^k D^{\gamma'} v - \frac{1}{i} k p x_j \langle x \rangle^{k-2} D^{\gamma'} v).$$

We may apply the inductive hypothesis to each of these terms and rewrite the result in the form (10.11); it is only necessary to check the order of the polynomials, and recall that $\langle x \rangle^2$ is a polynomial of degree 2. \square

Applying Lemma 10.6 to (10.10) gives (10.9), once negative powers of $\langle x \rangle$ are absorbed into the continuous functions. Then (10.8) follows from (10.9) and Leibniz's formula. \square