# SECOND PROBLEM SET FOR 18.155, FALL 2002
## DUE SEPTEMBER 24 IN CLASS OR 2-174.

Note that the books of Folland [1] and Rudin [2] cover most of the material in the early lectures. These problems are concerned with the Hahn decomposition theorem for measures – I will assume this in proving the Riesz representation theorem.

*Problem* 1. Let $(X, \mathcal{M})$ be a set with a $\sigma$-algebra. Let $\mu : \mathcal{M} \to \mathbb{R}$ be a finite measure in the sense that $\mu(\phi) = 0$ and for any $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ with $E_i \cap E_j = \phi$ for $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \tag{0.1}$$

with the series on the right *always* absolutely convergenct (i.e., this is part of the requirement on $\mu$). Define

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \tag{0.2}$$

for $E \in \mathcal{M}$, with the supremum over *all* measurable decompositions $E = \bigcup_{i=1}^{\infty} E_i$ with the $E_i$ disjoint. Show that $|\mu|$ is a finite, positive measure.

**Hint 1.** You must show that $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(A_i)$ if $\bigcup_i A_i = E$, $A_i \in \mathcal{M}$ being disjoint. Observe that if $A_j = \bigcup_l A_{jl}$ is a measurable decomposition of $A_j$ then together the $A_{jl}$ give a decomposition of $E$. Similarly, if $E = \bigcup_j E_j$ is any such decomposition of $E$ then $A_{jl} = A_j \cap E_l$ gives such a decomposition of $A_j$.

**Hint 2.** See [2] p. 117!

*Problem* 2. (Hahn Decomposition)

With assumptions as in Problem 1:

(1) Show that $\mu_+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu_- = \frac{1}{2}(|\mu| - \mu)$ are positive measures, $\mu = \mu_+ - \mu_-$. Conclude that the definition of a signed measure as simply the difference of two positive measures is the *same* as that in Problem 1.

(2) Show that $\mu_\pm$ so constructed are orthogonal in the sense that there is a set $E \in \mathcal{M}$ such that $\mu_-(E) = 0$, $\mu_+(X \setminus E) = 0$.

**Hint.** Use the definition of $|\mu|$ to show that for any $F \in \mathcal{M}$ and any $\epsilon > 0$ there is a subset $F' \in \mathcal{M}$, $F' \subset F$ such that $\mu_+(F') \geq \mu_+(F) - \epsilon$ and $\mu_-(F') \leq \epsilon$. Given $\delta > 0$ apply this result repeatedly (say with $\epsilon = 2^{-n}\delta$) to find a decreasing sequence of sets $F_1 = X$, $F_n \in \mathcal{M}$, $F_{n+1} \subset F_n$ such that $\mu_+(F_n) \geq \mu_+(F_{n-1}) - 2^{-n}\delta$ and $\mu_-(F_n) \leq 2^{-n}\delta$. Conclude that $G = \bigcap_n F_n$ has $\mu_+(G) \geq \mu_+(X) - \delta$ and $\mu_-(G) = 0$. Now let $G_m$ be chosen this way with $\delta = 1/m$. Show that $E = \bigcup_m G_m$ is as required.

*Problem* 3. Now suppose that $\mu$ is a finite, positive Radon measure on a locally compact metric space $X$ (meaning a finite positive Borel measure outer regular on Borel sets and inner regular on open sets). Show that $\mu$ is inner regular on all Borel sets and hence, given $\epsilon > 0$ and $E \in \mathcal{B}(X)$ there exist sets $K \subset E \subset U$ with $K$ compact and $U$ open such that $\mu(K) \geq \mu(E) - \epsilon$, $\mu(E) \geq \mu(U) - \epsilon$.

**Hint.** First take $U$ open, then use *its* inner regularity to find $K$ with $K' \Subset U$ and $\mu(K') \geq \mu(U) - \epsilon/2$. How big is $\mu(E \backslash K')$? Find $V \supset K' \backslash E$ with $V$ open and look at $K = K' \backslash V$.

*Problem* 4. Using Problem 3 show that if $\mu$ is a finite Borel measure on a locally compact metric space $X$ then the following three conditions are equivalent

(1) $\mu = \mu_1 - \mu_2$ with $\mu_1$ and $\mu_2$ both positive finite Radon measures.
(2) $|\mu|$ is a finite positive Radon measure.
(3) $\mu_+$ and $\mu_-$ are finite positive Radon measures.

## References

[1] G.B. Folland, *Real analysis*, Wiley, 1984.
[2] W. Rudin, *Real and complex analysis*, third edition ed., McGraw-Hill, 1987.