FIRST PROBLEM SET FOR 18.155, FALL 2002
DUE SEPTEMBER 17 IN CLASS OR 2-174.

Not that the books of Folland [1] and Rudin [2] cover most of the material in the early lectures.

Some of these questions are very easy.

Problem 1. Prove the Partition of unity lemma written up in class on Tuesday September 20:

Lemma 0.1. *[See also the lecture notes] Suppose U_i, $i = 1, \dots, N$ is ,a finite collection of open sets in a locally compact metric space and $K \Subset \bigcup_{i=1}^N U_i$ is a compact subset, then there exist continuous functions $f_i \in C(X)$ with $0 \le f_i \le 1$, $\operatorname{supp}(f_i) \Subset U_i$ and*

$$\sum_i f_i = 1 \text{ in a neighborhood of } K\,. \tag{0.1}$$

Hint(s). All functions here are supposed to be continuous, I just don't bother to keep on saying it.

(1) Recall, or check, that the local compactness of a metric space X means that for each point $x \in X$ there is an $\epsilon > 0$ such that the ball $\{y \in X; d(x, y) \le \delta\}$ is compact for $\delta \le \epsilon$.

(2) First do the case $n = 1$, so $K \Subset U$ is a compact set in an open subset.
   (a) Given $\delta > 0$, use the local compactness of X, to cover K with a finite number of compact closed balls of radius at most δ.
   (b) Deduce that if $\epsilon > 0$ is small enough then the set $\{x \in X; d(x, K) \le \epsilon\}$, where
   $$d(x, K) = \inf_{y \in K} d(x, y),$$
   is compact.
   (c) Show that $d(x, K)$, for K compact, is continuous.
   (d) Given $\epsilon > 0$ show that there is a continuous function $g_\epsilon : \mathbb{R} \longrightarrow [0, 1]$ such that $g_\epsilon(t) = 1$ for $t \le \epsilon/2$ and $g_\epsilon(t) = 0$ for $t > 3\epsilon/4$.
   (e) Show that $f = g_\epsilon \circ d(\cdot, K)$ satisfies the conditions for $n = 1$ if $\epsilon > 0$ is small enough.

(3) Prove the general case by induction over n.
   (a) In the general case, set $K' = K \cap U_1^{\complement}$ and show that the inductive hypothesis applies to K' and the U_j for $j > 1$; let f'_j, $j = 2, \dots, n$ be the functions supplied by the inductive assumption and put $f' = \sum_{j \ge 2} f'_j$.
   (b) Show that $K_1 = K \cap \{f' \le \frac{1}{2}\}$ is a compact subset of U_1.
   (c) Using the case $n = 1$ construct a function F for K_1 and U_1.
   (d) Use the case $n = 1$ again to find G such that $G = 1$ on K and $\operatorname{supp}(G) \Subset \{f' + F > \frac{1}{2}\}$.
   (e) Make sense of the functions
   $$f_1 = F \frac{G}{f' + F},\ f_j = f'_j \frac{G}{f' + F},\ j \ge 2$$

and show that they satisfies the inductive assumptions.

Problem 2. Show that σ-algebras are closed under countable intersections.

Problem 3. Show that if μ is a complete measure and $E \subset F$ where F is measurable and has measure 0 then $\mu(E) = 0$.

Problem 4. The Borel σ-algebra is the smallest σ-algebra on a topological space containing the open sets; the elements of the Borel σ-algebra are called Borel sets.

(1) Explain why such a smallest σ-algebra exists.
(2) Show that compact sets are Borel sets.

References

[1] G.B. Folland, *Real analysis*, Wiley, 1984.
[2] W. Rudin, *Real and complex analysis*, third edition ed., McGraw-Hill, 1987.