ANONYMOUS QUIZ FOR 18.155
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This anonymous quiz is just to help me judge the level at which I should begin. Please put an A, B or C in the margin on the left next to each question, where

- A Means that you know the answer straight away or how to prove the statement.
- B Means that you believe you could work it out in five or ten minutes.
- C Means that you suspect you don't know some necessary underlying results or do not understand the statement.

Note that there are several statements here that I *expect* you not to know or understand.

(1) Let $\mathcal{C}_0([-N,N])$ be the space of continuous functions on $\mathbb{R}$ which vanish outside $[-N,N] \subset \mathbb{R}$. Let $\mathcal{C}_\infty(\mathbb{R})$ be the space of bounded continuous functions on $\mathbb{R}$ with the supremum norm. Is the union $\bigcup_N \mathcal{C}_0([-N,N])$ dense in $\mathcal{C}_\infty(\mathbb{R})$?

(2) Let $\mathcal{C}_0([0,1])$ be the space of continuous functions on $[0,1]$ with supremum norm. Are there any continuous linear functionals $u : \mathcal{C}_0([0,1]) \longrightarrow \mathbb{C}$ such that $u(fg) = u(f)u(g)$ for all $f, g \in \mathcal{C}_0([0,1])$, where $fg(x) = f(x)g(x)$?

(3) Let $L^1([0,1])$ and $L^2([0,1])$ be the Lebesgue spaces on $[0,1]$. What exactly is an element of each these spaces? What are their standard norms?

(4) Which of $L^1([0,1])$ and $L^2([0,1])$ is a Hilbert space?

(5) What are all the continuous linear functionals $u : L^1([0,1]) \longrightarrow \mathbb{C}$ such that $u(fg) = u(f)u(g)$ for all $f, g \in L^2([0,1])$.

(6) Let $u : \{(x,y); x^2 + y^2 < 1\} \longrightarrow \mathbb{C}$ be a once differentiable function on the open unit ball which satisfies

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = 0 \text{ in } x^2 + y^2 < 1.$$

Why is it true that u is infinitely differentiable?

(7) What functions are there as in the previous question which satisfy in addition $\frac{\partial^k u}{\partial x^k}(0,0) = 0$ for all k?

(8) Every twice differentiable solution of the wave equation in two variables, $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ in $\mathbb{R}^2$ is of the form $u_1(x+t) + u_2(x-t)$ for two twice differentiable functions of one variable.

(9) There is no smooth map $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ which is surjective.

(10) For any sequence of real numbers a_j, $j = 0, 1, \ldots$, there is a smooth function $u : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\frac{d^j u}{dx^j}(0) = a_j$ for all j.

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