NINTH ASSIGNMENT, DUE NOVEMBER 20 IN CLASS
18.155 FALL 2001

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Problem 1. Work out the elementary behavior of the heat equation.

i) Show that the function on $\mathbb{R} \times \mathbb{R}^n$, for $n \geq 1$,

$$F(t,x) = \begin{cases} t^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is measurable, bounded on the any set $\{|(t,x)| \geq R\}$ and is integrable on $\{|(t,x)| \leq R\}$ for any $R > 0$.

ii) Conclude that F defines a tempered distibution on $\mathbb{R}^{n+1}$.

iii) Show that F is $\mathcal{C}^\infty$ outside the origin.

iv) Show that F satisfies the heat equation

$$(\partial_t - \sum_{j=1}^n \partial^2_{x_j})F(t,x) = 0 \text{ in } (t,x) \neq 0.$$

v) Show that F satisfies

(1)
$$F(s^2t, sx) = s^{-n}F(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1})$$

where the left hand side is defined by duality "$F(s^2t, sx) = F_s$" where

$$F_s(\phi) = s^{-n-2}F(\phi_{1/s}),\ \phi_{1/s}(t,x) = \phi(\frac{t}{s^2}, \frac{x}{s}).$$

vi) Conclude that

$$(\partial_t - \sum_{j=1}^n \partial^2_{x_j})F(t,x) = G(t,x)$$

where $G(t,x)$ satisfies

(2)
$$G(s^2t, sx) = s^{-n-2}G(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1})$$

in the same sense as above and has support at most $\{0\}$.

vii) Hence deduce that

(3)
$$(\partial_t - \sum_{j=1}^n \partial^2_{x_j})F(t,x) = c\delta(t)\delta(x)$$

for some real constant c.

Hint: Check which distributions with support at $(0,0)$ satisfy (2).

viii) If $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$ show that $u = F \star \psi$ satisfies

(4)
$$u \in \mathcal{C}^\infty(\mathbb{R}^{n+1}) \text{ and } \sup_{x\in\mathbb{R}^n,\ t\in[-S,S]} (1+|x|)^N |D^\alpha u(t,x)| < \infty\ \forall\ S > 0, \alpha \in \mathbb{N}^{n+1},\ N.$$

ix) Supposing that u satisfies (4) and is a real-valued solution of

$$(\partial_t - \sum_{j=1}^{n} \partial_{x_j}^2) u(t,x) = 0$$

in $\mathbb{R}^{n+1}$, show that

$$v(t) = \int_{\mathbb{R}^n} u^2(t,x)$$

is a non-increasing function of t.

Hint: Multiply the equation by u and integrate over a slab $[t_1, t_2] \times \mathbb{R}^n$.

x) Show that c in (3) is non-zero by arriving at a contradiction from the assumption that it is zero. Namely, show that if $c = 0$ then u in viii) satisfies the conditions of ix) and also vanishes in $t < T$ for some T (depending on ψ). Conclude that $u = 0$ for all ψ. Using properties of convolution show that this in turn implies that $F = 0$ which is a contradiction.

xi) So, finally, we know that $E = \frac{1}{c} F$ is a fundamental solution of the heat operator which vanishes in $t < 0$. Explain why this allows us to show that for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ there is a solution of

$$(\partial_t - \sum_{j=1}^{n} \partial_{x_j}^2) u = \psi, \ u = 0 \text{ in } t < T \text{ for some } T. \tag{5}$$

What is the largest value of T for which this holds?

xii) Can you give a heuristic, or indeed a rigorous, explanation of why

$$c = \int_{\mathbb{R}^n} \exp(-\frac{|x|^2}{4}) dx?$$

xiii) Explain why the argument we used for the wave equation to show that there is *only one* solution, $u \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, of (5) does not apply here. (Indeed such uniqueness does not hold without some growth assumption on u.)

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