EIGHT ASSIGNMENT, DUE NOVEMBER 6 IN CLASS
18.155 FALL 2001

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Problem 1. Wavefront set computations and more – all pretty easy, especially if you use results from class.

i) Compute $\mathrm{WF}(\delta)$ where $\delta \in \mathcal{S}'(\mathbb{R}^n)$ is the Dirac delta function at the origin.
ii) Compute $\mathrm{WF}(H(x))$ where $H(x) \in \mathcal{S}'(\mathbb{R})$ is the Heaviside function
$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}.$$
Hint: D_x is elliptic in one dimension, hit H with it.
iii) Compute $\mathrm{WF}(E)$, $E = iH(x_1)\delta(x')$ which is the Heaviside in the first variable on $\mathbb{R}^n$, $n > 1$, and delta in the others.
iv) Show that $D_{x_1}E = \delta$, so E is a fundamental solution of D_{x_1}.
v) If $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ show that $u = E \star f$ solves $D_{x_1}u = f$.
vi) What does our estimate on $\mathrm{WF}(E \star f)$ tell us about $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$?

Problem 2. The wave equation in two variables (or one spatial variable).

i) Recall that the Riemann function
$$E(t,x) = \begin{cases} -\frac{1}{4} & \text{if } t > x \text{ and } t > -x \\ 0 & \text{otherwise} \end{cases}$$
is a fundamental solution of $D_t^2 - D_x^2$ (check my constant).
ii) Find the singular support of E.
iii) Write the Fourier transform (dual) variables as τ, ξ and show that
$$\mathrm{WF}(E) \subset \{0\} \times \mathbb{S}^1 \cup \{(t,x,\tau,\xi); x = t > 0 \text{ and } \xi + \tau = 0\} \\ \cup \{(t,x,\tau,\xi); -x = t > 0 \text{ and } \xi = \tau\}.$$
iv) Show that if $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^2)$ then $u = E \star f$ satisfies $(D_t^2 - D_x^2)u = f$.
v) With u defined as in iv) show that
$$\mathrm{supp}(u) \subset \{(t,x);\ \exists\ (t',x') \in \mathrm{supp}(f) \text{ with } t' + x' \le t + x \text{ and } t' - x' \le t - x\}.$$
vi) Sketch an illustrative example of v).
vii) Show that, still with u given by iv),
$$\mathrm{sing\,supp}(u) \subset \{(t,x);\ \exists\ (t',x') \in \mathrm{sing\,supp}(f) \text{ with} \\ t \ge t' \text{ and } t + x = t' + x' \text{ or } t - x = t' - x'\}.$$
viii) Bound $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$.

Problem 3. A little uniqueness theorems. Suppose $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ recall that the Fourier transform $\hat{u} \in \mathcal{C}^\infty(\mathbb{R}^n)$. Now, suppose $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies $P(D)u = 0$ for some non-trivial polynomial P, show that $u = 0$.

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