SEVENTH ASSIGNMENT, DUE OCTOBER 30 IN CLASS 18.155 FALL 2001

RICHARD MELROSE

Problem 1. Distributions of compact support.

i) Recall the definition of the support of a distribution, defined in terms of its complement

$$\mathbb{R}^n \setminus \operatorname{supp}(u) = \left\{p \in \mathbb{R}^n; \exists\, U \subset \mathbb{R}^n, \text{ open, with } p \in U \text{ such that } u\big|_U = 0\right\}$$

ii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}(u) \cap \operatorname{supp}(\phi) = \emptyset$$

then $u(\phi) = 0$.

iii) Consider the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of all smooth functions on $\mathbb{R}^n$, without restriction on supports. Show that for each N

$$\|f\|_{(N)} = \sup_{|\alpha| \le N,\ |x| \le N} |D^\alpha f(x)|$$

is a seminorn on $\mathcal{C}^\infty(\mathbb{R}^n)$ (meaning it satisfies $\|f\| \ge 0$, $\|cf\| = |c|\|f\|$ for $c \in \mathbb{C}$ and the triangle inequality but that $\|f\| = 0$ does not necessarily imply that $f = 0$.

iv) Show that $\mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ is dense in the sense that for each $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ there is a sequence f_n in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\|f - f_n\|_{(N)} \to 0$ for each N.

v) Let $\mathcal{E}'(\mathbb{R}^n)$ temporarily (or permanantly if you prefer) denote the dual space of $\mathcal{C}^\infty(\mathbb{R}^n)$ (which is also written $\mathcal{E}(\mathbb{R}^n)$), that is, $v \in \mathcal{E}'(\mathbb{R}^n)$ is a linear map $v : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C}$ which is continuous in the sense that for some N

$$|v(f)| \le C\|f\|_{(N)}\ \forall\ f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{1}$$

Show that such a v 'is' a distribution and that the map $\mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n)$ is injective.

vi) Show that if $v \in \mathcal{E}'(\mathbb{R}^n)$ satisfies (1) and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ has $f = 0$ in $|x| < N + \epsilon$ for some $\epsilon > 0$ then $v(f) = 0$.

vii) Conclude that each element of $\mathcal{E}'(\mathbb{R}^n)$ has compact support when considered as an element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$.

viii) Show the converse, that each element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ with compact support is an element of $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and hence conclude that $\mathcal{E}'(\mathbb{R}^n)$ 'is' the space of distributions of compact support.

I will denote the space of distributions of compact support by $\mathcal{C}_c^{-\infty}(\mathbb{R})$.

Problem 2. Hypoellipticity of the heat operator $H = iD_t + \Delta = iD_t + \sum_{j=1}^{n} D_{x_j}^2$ on $\mathbb{R}^{n+1}$.

(1) Using τ to denote the 'dual variable' to t and $\xi \in \mathbb{R}^n$ to denote the dual variables to $x \in \mathbb{R}^n$ observe that $H = p(D_t, D_x)$ where $p = i\tau + |\xi|^2$.
(2) Show that $|p(\tau,\xi)| > \frac{1}{2}\left(|\tau| + |\xi|^2\right)$.
(3) Use an inductive argument to show that, in $(\tau,\xi) \neq 0$ where it makes sense,

$$D_\tau^k D_\xi^\alpha \frac{1}{p(\tau,\xi)} = \sum_{j=1}^{|\alpha|} \frac{q_{k,\alpha,j}(\xi)}{p(\tau,\xi)^{k+j+1}} \tag{2}$$

where $q_{k,\alpha,j}(\xi)$ is a polynomial of degree (at most) $2j - |\alpha|$.
(4) Conclude that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ is identically equal to 1 in a neighbourhood of 0 then the function

$$g(\tau,\xi) = \frac{1-\phi(\tau,\xi)}{i\tau + |\xi|^2}$$

is the Fourier transform of a distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{sing\,supp}(F) \subset \{0\}$. [Remember that $\operatorname{sing\,supp}(F)$ is the complement of the largest open subset of $\mathbb{R}^n$ the restriction of F to which is smooth].
(5) Show that F is a parametrix for the heat operator.
(6) Deduce that $iD_t + \Delta$ is *hypoelliptic* – that is, if $U \subset \mathbb{R}^n$ is an open set and $u \in \mathcal{C}^{-\infty}(U)$ satisfies $(iD_t + \Delta)u \in \mathcal{C}^\infty(U)$ then $u \in \mathcal{C}^\infty(U)$.
(7) Show that $iD_t - \Delta$ is also hypoelliptic.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu