FIFTH ASSIGNMENT, DUE OCTOBER 16 IN CLASS
18.155 FALL 2001

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Problem 1. Let $X \subset \mathbb{R}^n$ be open and denote by $\mathcal{C}^0_{\mathrm{c}}(X;\mathbb{R})$ the space of real-valued continuous functions on X which vanish outside some compact subset of X. Show that each $f \in \mathcal{C}^0_{\mathrm{c}}(X;\mathbb{R})$ can be written uniquely in the form $f = f_+ - f_-$ where $0 \le f_\pm \in \mathcal{C}^0_{\mathrm{c}}(X;\mathbb{R})$ and $f_+(x)\cdot f_-(x) = 0 \ \forall \ x \in X$.

Problem 2. Let u be a measure on $X \subset \mathbb{R}^n$, open, in the sense that $u : \mathcal{C}^0_{\mathrm{c}}(X;\mathbb{R}) \longrightarrow \mathbb{R}$ is linear and continuous—meaning that for each $K \Subset X$ there exists C such that

$$|u(f)| \le C \sup_X |f(x)| \ \forall \ f \in \mathcal{C}^0_{\mathrm{c}}(X) \text{ with } \operatorname{supp}(f) \subset K\,.$$

Show that defining

$$u_+(f) = \sup\{u(g);\, g \in \mathcal{C}^0_{\mathrm{c}}(X),\ 0 \le g(x) \le f(x) \ \forall \ x \in X\},\ \text{if } f \ge 0\,,$$

fixes a measure in the same sense and that

$$u_-(f) \ge 0 \text{ when } f \ge 0,\ \text{where } u_- = -u + u_+\,.$$

Problem 3. Suppose $u : \mathcal{C}^0_{\mathrm{c}}(X;\mathbb{R}) \longrightarrow \mathbb{R}$ is as in the previous question and is positive, i.e., $u(f) \ge 0$ if $f \ge 0$. Define

$$\mu(E) = \sup\{u(f);\, f \in \mathcal{C}^0_{\mathrm{c}}(E),\ 0 \le f \le 1\} \in [0,\infty]$$

for each *open* set $E \subset X$ and in general

$$\mu^*(A) = \inf\{\mu(E);\, A \subset E \subset X,\ E \text{ open}\}\,.$$

Show that μ^* is a countably sub-additive function (an outer measure), $\mu^*(\emptyset) = 0$,

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu^*(A_j)\,.$$

Problem 4. Go through the proof of Caratheodory's theorem in this more general case, showing that $\mu = \mu^*$ is a countably additive measure on the σ-algebra of all μ-measurable sets, where μ-measurability (of B) is the condition that

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^{\complement}) \ \forall \ A \subset X\,.$$

Problem 5. Read through the proof of the Riesz representation theorem from the notes on the web.

Problem 6. Show that

$$e^x \cos(e^x) \in \mathcal{S}'(\mathbb{R})\,.$$

Hint: Find a primitive.

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