FOURTH ASSIGNMENT, DUE OCTOBER 2 IN CLASS
18.155 FALL 2001

RICHARD MELROSE

Note that the problem should from now on be available on Tuesday, still due he following Tuesday in class. This is by request, to give you a little more time. At least some of the questions will be based on that Tuesday's class. All three questions here are standard results for Lebesgue. If you want hints, look at the HW4H.* files! The first of these questions is the sort of annoying thing that comes up in elementary measure theory (I did not want to do it in lecture). The rest of the problems are fairly straightforward.

Problem 1. Show that for any set $G \subset \mathbb{R}^n$

$$v^*(G) = \inf \sum_{i=1}^{\infty} v(A_i)$$

where the infimum is taken over coverings of G by rectangular sets (products of intervals).

Problem 2. Show that a σ-algebra is closed under countable intersections.

Problem 3. Show that compact sets are Lebesgue measurable and have finite volume and also show the inner regularity of the Lebesgue measure on open sets, that is if E is open then

$$v(E) = \sup\{v(K); K \subset E,\ K \text{ compact}\}. \tag{1}$$

Problem 4. Show that a set $B \subset \mathbb{R}^n$ is Lebesgue measurable if and only if

$$v^*(E) = v^*(E \cap B) + v^*(E \cap B^{\complement})\ \forall \text{ open } E \subset \mathbb{R}^n.$$

[The definition is this for all $E \subset \mathbb{R}^n$.]

Problem 5. Show that a real-valued continuous function $f : U \longrightarrow \mathbb{R}$ on an open set, is Lebesgue measurable, in the sense that $f^{-1}(I) \subset U \subset \mathbb{R}^n$ is measurable for each interval I.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu