THIRD ASSIGNMENT, DUE SEPTEMBER 25 IN CLASS
18.155 FALL 2001

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Special note: Full marks may be achieved without doing the last three parts of the third problem.

These three problems are all about homogeneous distributions on the line, extending various things I did in class. We observed that

$$x_+^z = \begin{cases} \exp(z \log x) & x > 0 \\ 0 & x \le 0 \end{cases}$$

is a continuous function on $\mathbb{R}$ if $\operatorname{Re} z > 0$ and is differentiable if $\operatorname{Re} z > 1$ and then satisfies

$$\frac{d}{dx} x_+^z = z x_+^{z-1}.$$

We used this to define

$$x_+^z = \frac{1}{z+k}\frac{1}{z+k-1}\cdots\frac{1}{z+1}\frac{d^k}{dx^k}x_+^{z+k} \text{ if } z \in \mathbb{C} \setminus -\mathbb{N}. \tag{1}$$

Problem 1. [Hadamard regularization]

i) Show that (1) just means that for each $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$x_+^z(\phi) = \frac{(-1)^k}{(z+k)\cdots(z+1)} \int_0^\infty \frac{d^k\phi}{dx^k}(x) x^{z+k} dx, \ \operatorname{Re} z > -k, \ z \notin -\mathbb{N}.$$

ii) Use integration by parts to show that

$$x_+^z(\phi) = \lim_{\epsilon \downarrow 0} \left[\int_\epsilon^\infty \phi(x) x^z dx - \sum_{j=1}^k C_j(\phi) \epsilon^{z+j} \right], \ \operatorname{Re} z > -k, \ z \notin -\mathbb{N} \tag{2}$$

for certain constants $C_j(\phi)$ which you should give explicitly. [This is called Hadamard regularization after Jacques Hadamard, feel free to look at his classic book [1].]

iii) Assuming that $-k+1 \ge \operatorname{Re} z > -k$, $z \ne -k+1$, show that there can only be one set of the constants with $j < k$ (for each choice of $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$) such that the limit in (2) exists.

iiv) Use ii), and maybe iii), to show that

$$\frac{d}{dx} x_+^z = z x_+^{z-1} \text{ in } \mathcal{C}^{-\infty}(\mathbb{R}) \ \forall \ z \notin -\mathbb{N}_0 = \{0, 1, \dots\}.$$

v) Similarly show that $x x_+^z = x_+^{z+1}$ for all $z \notin -\mathbb{N}$.

vi) Show that $x_+^z = 0$ in $x < 0$ for all $z \notin -\mathbb{N}$. (Duh.)

Problem 2. [Null space of $x\frac{d}{dx} - z$]

i) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ then $\tilde{u}(\phi) = u(\tilde{\phi})$, where $\tilde{\phi}(x) = \phi(-x)\ \forall\ \phi \in \mathcal{C}^\infty_c(\mathbb{R})$, defines an element of $\mathcal{C}^{-\infty}(\mathbb{R})$. What is $\tilde{u}$ if $u \in \mathcal{C}^0(\mathbb{R})$? Compute $\widetilde{\delta_0}$.

ii) Show that $\frac{d}{dx}\tilde{u} = -\widetilde{\frac{d}{dx}u}$.

iii) Define $x^z_- = \widetilde{x^z_+}$ for $z \notin -\mathbb{N}$ and show that $\frac{d}{dx}x^z_- = -zx^{z-1}_-$ and $xx^z_- = -x^{z+1}_-$.

iv) Suppose that $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfies the distributional equation $(x\frac{d}{dx} - z)u = 0$ (meaning of course, $x\frac{du}{dx} = zu$ where z is a constant). Show that

$$u\big|_{x>0} = c_+ x^z_-\big|_{x>0} \text{ and } u\big|_{x<0} = c_- x^z_-\big|_{x<0}$$

for some constants $c_\pm$. Deduce that $v = u - c_+x^z_+ - c_-x^z_-$ satisfies

$$(x\frac{d}{dx} - z)v = 0 \text{ and } \operatorname{supp}(v) \subset \{0\}. \tag{3}$$

v) Show that for each $k \in \mathbb{N}$, $(x\frac{d}{dx} + k + 1)\frac{d^k}{dx^k}\delta_0 = 0$.

vi) Using the *fact* that any $v \in \mathcal{C}^{-\infty}(\mathbb{R})$ with $\operatorname{supp}(v) \subset \{0\}$ is a finite sum of constant multiples of the $\frac{d^k}{dx^k}\delta_0$, show that, for $z \notin -\mathbb{N}$, the only solution of (3) is $v = 0$.

vii) Conclude that for $z \notin -\mathbb{N}$

$$\left\{u \in \mathcal{C}^{-\infty}(\mathbb{R}); (x\frac{d}{dx} - z)u = 0\right\} \tag{4}$$

is a two-dimensional vector space.

Problem 3. [Negative integral order] To do the same thing for negative integral order we need to work a little differently. Fix $k \in \mathbb{N}$.

i) We define *weak convergence* of distributions by saying $u_n \to u$ in $\mathcal{C}^\infty_c(X)$, where $u_n, u \in \mathcal{C}^{-\infty}(X)$, $X \subset \mathbb{R}^n$ being open, if $u_n(\phi) \to u(\phi)$ for each $\phi \in \mathcal{C}^\infty_c(X)$. Show that $u_n \to u$ implies that $\frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ for each $j = 1, \dots, n$ and $fu_n \to fu$ if $f \in \mathcal{C}^\infty(X)$.

ii) Show that $(z + k)x^z_+$ is weakly continuous as $z \to -k$ in the sense that for any sequence $z_n \to -k$, $z_n \notin -\mathbb{N}$, $(z_n + k)x^{z_n}_+ \to v_k$ where

$$v_k = \frac{1}{-1} \cdots \frac{1}{-k+1}\frac{d^{k+1}}{dx^{k+1}}x_+,\ x_+ = x^1_+.$$

iii) Compute v_k, including the constant factor.

iv) Do the same thing for $(z + k)x^z_-$ as $z \to -k$.

v) Show that there is a linear combination $(k + z)(x^z_+ + c(k)x^z_-)$ such that as $z \to -k$ the limit is zero.

vi) If you get this far, show that in fact $x^z_+ + c(k)x^z_-$ also has a weak limit, u_k, as $z \to -k$. [This may be the hardest part.]

vii) Show that this limit distribution satisfies $(x\frac{d}{dx} + k)u_k = 0$.

viii) Conclude that (4) does in fact hold for $z \in -\mathbb{N}$ as well. [There are still some things to prove to get this.]

References

[1] J. Hadamard, *Le problème de Cauchy et les èquatons aux dérivées partielles linéaires hyperboliques*, Hermann, Paris, 1932.

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