SECOND ASSIGNMENT WITH SOLUTIONS
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In the main these questions form theorems in Hörmander's book [1], so the proofs are available there. I suggest that you try to work them out on your own and in any case I expect written proofs, even if you need to get the idea from the book. Of course, at the very least, you will have to translate the notation.

Problem 1. [Hörmander, Theorem 3.1.4] Let $I \subset \mathbb{R}$ be an open, non-empty interval.

i) Show (you may use results from class) that there exists $\psi \in \mathcal{C}_c^\infty(I)$ with $\int_{\mathbb{R}} \psi(x)ds = 1$.

Solution. We showed in class that there exists $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ which is strictly positive on $(0,1)$ and vanishes on $\mathbb{R}\setminus(0,1)$. If I is a non-empty open interval it contains an interval $[\bar{x}-\epsilon, \bar{x}+\epsilon]$ for some $\bar{x} \in \mathbb{R}$ and $\epsilon > 0$. Consider $\psi(x) = C\phi((x-\bar{x})/\epsilon)$ where $C > 0$ is to be chosen. The properties of ϕ imply that $\psi \in \mathcal{C}_c^\infty(I)$ and that $\int_{\mathbb{R}} \psi = 1/C$, $C > 0$. This value of C make the integral of ψ equal 1 as desired. □

ii) Show that any $\phi \in \mathcal{C}_c^\infty(I)$ may be written in the form

$$\phi = \tilde{\phi} + c\psi,\ c \in \mathbb{C},\ \tilde{\phi} \in \mathcal{C}_c^\infty(I) \text{ with } \int_{\mathbb{R}} \tilde{\phi} = 0.$$

Solution. Set $c = \int_I \phi$ and observe that $\tilde{\phi} = \phi - c\psi$ has vanishing integral. □

iii) Show that if $\tilde{\phi} \in \mathcal{C}_c^\infty(I)$ and $\int_{\mathbb{R}} \tilde{\phi} = 0$ then there exists $\mu \in \mathcal{C}_c^\infty(I)$ such that $\frac{d\mu}{dx} = \tilde{\phi}$ in I.

Solution. If we consider $\mu(x) = \int_{-\infty}^x \tilde{\phi}(s)ds$ then μ is infinitely differentiable since $d\mu(x)/dx = \tilde{\phi}(x)$ by the fundamental theorem of calculus. Certainly $\mu(x) = 0$ if $x < \inf I'$, where I' is a compact interval outside which $\tilde{\phi}$ vanishes. The assumption that $\int_I \tilde{\phi}(x)dx = \int_{I'} \tilde{\phi}(x)dx = 0$ means that $\mu(x) = 0$ for $x > \sup I'$ so $\mu \in \mathcal{C}_c^\infty(I)$. □

iv) Suppose $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = 0$, i.e.

$$u(-\frac{d\phi}{dx}) = 0\ \forall\ \phi \in \mathcal{C}_c^\infty(I), \tag{1}$$

show that $u = c$ for some constant c.

Solution. Using ii) and iii) we may write each element $\phi \in \mathcal{C}_c^\infty(I)$ in the form $\phi = c\psi + d\mu(x)/dx$ where $\psi \in \mathcal{C}_c^\infty(I)$ is fixed and $\mu \in \mathcal{C}_c^\infty(I)$ depends on ϕ. Then the definition of the vanishing of the derivative in 1 shows that

$$u(\phi) = u(c\psi + d\mu/dx) = cu(\psi) - u(-d\mu/dx) = cu(\psi) = \int_I C\phi(x)dx$$

where we have used the value of c and set $C = u(\psi)$. The last integral is the definition of the (constant) function C as a distribution on I. Thus $u = C$ is an equality between distributions. □

v) Suppose that $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = c$, for some constant c, show that $u = cx + d$ for some $d \in \mathbb{C}$.

Solution. Consider the continuous function cx. This defines a distribution, which we just denote the same way. The definition of the derivative means that $d(cx)/dx = c$ (as you would expect). We proved this in class, it is the integration by parts formula

$$\frac{d(cx)}{dx}(\phi) = cx(-\frac{d\phi}{dx}) = -\int_{\mathbb{R}} cx\frac{d\phi}{dx}dx = c(\phi).$$

Now, if we consider the difference $v = u - cx$ it is a distribution on I which satisfies $dv/dx = 0$, so by iv) it is equal to a constant, d. That is $u - cx = d$, or $u = cx + d$. □

Problem 2. [Hörmander Theorem 3.1.16]

i) Use Taylor's formula to show that there is a fixed $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ can be written in the form

$$\phi = c\psi + \sum_{j=1}^n x_j\psi_j \tag{2}$$

where $c \in \mathbb{C}$ and the $\psi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ depend on ϕ.

Solution. By Taylor's formula, or theorem, any infinitely differentiable function which vanishes at 0 can be written in the form

$$\phi = \sum_{j=1}^n x_j u_j \tag{3}$$

where the u_j are also infinitely differeniable. Choosing $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, as in class, with $\psi(0) = 1$ we can apply this to $\phi - c\psi$ where $c = \phi(0)$. The problem is that the u_j need not have compact support. However, we can choose another function (depending on ϕ), $\psi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ which is equal to 1 on the support of ϕ and the support of ψ. Thus $\phi\psi' = \phi$ and $\psi\psi' = \psi$. Then 3 gives the desired 2 where $\psi_j - u_j\psi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. □

ii) Recall that δ_0 is the distribution defined by

$$\delta_0(\phi) = \phi(0) \ \forall \ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n);$$

explain why $\delta_0 \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$.

Solution. Certainly δ_0 so defined is a linear functional and it is continuous since

$$|\delta_0(\phi)| \le \sup_{\mathbb{R}^n} |\phi| = \|\phi\|_0.$$

□

iii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $u(x_j\phi) = 0$ for all $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and $j = 1, \dots, n$ then $u = c\delta_0$ for some $c \in \mathbb{C}$.

Solution. We can use 2 to evaluate

$$u(\phi) = u(c\psi) + \sum_{j=1}^n u(x_j\psi_j) = C\phi(0)$$

where the given identity means that the sum is zero and we write $C = u(\psi)$ for a fixed constant and note that $c = \phi(0)$. This just means $u = C\delta_0$. □

iv) Define the 'Heaviside function'

$$H(\phi) = \int_0^\infty \phi(x)dx \ \forall \ \phi \in \mathcal{C}^\infty_c(\mathbb{R});$$

show that $H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

Solution. The linearity of the integral shows that H is a linear functional on $\mathcal{C}^\infty_c(\mathbb{R})$. The basic integral estimate

$$|H(\phi)| \le L \sup|\phi| \text{ if } \phi(x) = 0 \text{ in } |x| > L$$

shows that it is continuous. □

v) Compute $\frac{d}{dx}H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

Solution. By definition

$$\frac{dH}{dx}(\phi) = -H(\frac{d\phi}{dx}) = -\int_0^\infty \frac{d\phi}{dx}dx = \phi(0)$$

where we have used the fundamental theorem of calculus. Thus

$$\frac{dH}{dx} = \delta_0. \tag{4}$$

□

Problem 3. Using Problems 1 and 2, find all $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfying the differential equation

$$x\frac{du}{dx} = 0 \text{ in } \mathbb{R}.$$

Solution. By definition $v = \frac{du}{dx}$ is a distribution which satisfies $xv = 0$. Problem 2, part iii), in one dimension shows that $v = c\delta_0$. Thus $\frac{du}{dx} = c\delta_0$. By Problem 2, part v), $dH/dx = \delta$. Thus $\tilde{u} = u - cH$ satisfies $d\tilde{u}/dx = 0$. By Problem 1, part iv), we conclude that $v = d$ is constant. Thus the only distributional solutions of $x\frac{du}{dx} = 0$ are

$$u = cH + d, \ c \text{ and } d \text{ constants.}$$

This means that u is constant in $x < 0$ and in $x > 0$, with possibly different values, but it means more than this as regards its 'behaviour at 0'. □

References

[1] L. Hörmander, *The analysis of linear partial differential operators*, vol. 1, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

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