

**SECOND ASSIGNMENT, DUE SEPTEMBER 18 IN CLASS  
18.155 FALL 2001**

RICHARD MELROSE

In the main these questions form theorems in Hörmander's book [1], so the proofs are available there. I suggest that you try to work them out on your own and in any case I expect written proofs, even if you need to get the idea from the book. Of course, at the very least, you will have to translate the notation.

*Problem 1.* [Hörmander, Theorem 3.1.4] Let  $I \subset \mathbb{R}$  be an open, non-empty interval.

i) Show (you may use results from class) that there exists  $\psi \in \mathcal{C}_c^\infty(I)$  with  $\int_{\mathbb{R}} \psi(x) dx = 1$ .

ii) Show that any  $\phi \in \mathcal{C}_c^\infty(I)$  may be written in the form

$$\phi = \tilde{\phi} + c\psi, \quad c \in \mathbb{C}, \quad \tilde{\phi} \in \mathcal{C}_c^\infty(I) \text{ with } \int_{\mathbb{R}} \tilde{\phi} = 0.$$

iii) Show that if  $\tilde{\phi} \in \mathcal{C}_c^\infty(I)$  and  $\int_{\mathbb{R}} \tilde{\phi} = 0$  then there exists  $\mu \in \mathcal{C}_c^\infty(I)$  such that  $\frac{d\mu}{dx} = \tilde{\phi}$  in  $I$ .

iv) Suppose  $u \in \mathcal{C}^{-\infty}(I)$  satisfies  $\frac{du}{dx} = 0$ , i.e.

$$u\left(-\frac{d\phi}{dx}\right) = 0 \quad \forall \phi \in \mathcal{C}_c^\infty(I),$$

show that  $u = c$  for some constant  $c$ .

v) Suppose that  $u \in \mathcal{C}^{-\infty}(I)$  satisfies  $\frac{du}{dx} = c$ , for some constant  $c$ , show that  $u = cx + d$  for some  $d \in \mathbb{C}$ .

*Problem 2.* [Hörmander Theorem 3.1.16]

i) Use Taylor's formula to show that there is a fixed  $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  such that any  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  can be written in the form

$$\phi = c\psi + \sum_{j=1}^n x_j \psi_j$$

where  $c \in \mathbb{C}$  and the  $\psi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  depend on  $\phi$ .

ii) Recall that  $\delta_0$  is the distribution defined by

$$\delta_0(\phi) = \phi(0) \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n);$$

explain why  $\delta_0 \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ .

iii) Show that if  $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$  and  $u(x_j \phi) = 0$  for all  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  and  $j = 1, \dots, n$  then  $u = c\delta_0$  for some  $c \in \mathbb{C}$ .

iv) Define the 'Heaviside function'

$$H(\phi) = \int_0^\infty \phi(x) dx \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R});$$

show that  $H \in \mathcal{C}^{-\infty}(\mathbb{R})$ .

v) Compute  $\frac{d}{dx} H \in \mathcal{C}^{-\infty}(\mathbb{R})$ .

*Problem 3.* Using Problems 1 and 2, find all  $u \in C^{-\infty}(\mathbb{R})$  satisfying the differential equation

$$x \frac{du}{dx} = 0 \text{ in } \mathbb{R}.$$

## REFERENCES

- [1] L. Hörmander, *The analysis of linear partial differential operators*, vol. 1, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
*E-mail address:* `rbm@math.mit.edu`