FIRST ASSIGNMENT, DUE SEPTEMBER 11 IN CLASS 18.155 FALL 2001

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In the solutions of these problems I am looking for precise statements and clear, succinct proofs. Some of these problem may involve things you do not know – of course, as with the anonymous quiz, I am simply trying to check your level of knowledge so that I can adjust the course as necessary. No weight, in terms of final grade, will be given to this assignment but please do it anyway and not anonymously this time.

Problem 1

[Taylor's theorem]. Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued function which is k times continuously differentiable. Prove that there is a polynomial p and a continuous function v such that

$$u(x) = p(x) + v(x) \text{ where } \lim_{|x|\downarrow 0} \frac{|v(x)|}{|x|^k} = 0.$$

Problem 2

Let $\mathcal{C}(\mathbb{B}^n)$ be the space of continuous functions on the (closed) unit ball, $\mathbb{B}^n = \{x \in \mathbb{R}^n; |x| \le 1\}$. Let $\mathcal{C}_0(\mathbb{B}^n) \subset \mathcal{C}(\mathbb{B}^n)$ be the subspace of functions which vanish at each point of the boundary and let $\mathcal{C}(\mathbb{S}^{n-1})$ be the space of continuous functions on the unit sphere. Show that inclusion and restriction to the boundary gives a short exact sequence

$$\mathcal{C}_0(\mathbb{B}^n) \hookrightarrow \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathcal{C}(\mathbb{S}^{n-1})$$

(meaning the first map is injective, the second is surjective and the image of the first is the null space of the second.)

Problem 3

[Measures] A measure on the ball is a continuous linear functional $\mu : \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathbb{R}$ where continuity is with respect to the supremum norm, i.e. there must be a constant C such that

$$|\mu(f)| \le C \sup_{x\in\mathbb{R}^n} |f(x)| \ \forall\ f \in \mathcal{C}(\mathbb{B}^n).$$

Let $M(\mathbb{B}^n)$ be the linear space of such measures. The space $M(\mathbb{S}^{n-1})$ of measures on the sphere is defined similarly. Describe an injective map

$$M(\mathbb{S}^{n-1}) \longrightarrow M(\mathbb{B}^n).$$

Can you define another space so that this can be extended to a short exact sequence?

Problem 4

Show that the Riemann integral defines a measure

$$\mathcal{C}(\mathbb{B}^n) \ni f \longmapsto \int_{\mathbb{B}^n} f(x)dx. \tag{1}$$

Problem 5

If $g \in \mathcal{C}(\mathbb{B}^n)$ and $\mu \in M(\mathbb{B}^n)$ show that $g\mu \in M(\mathbb{B}^n)$ where $(g\mu)(f) = \mu(fg)$ for all $f \in \mathcal{C}(\mathbb{B}^n)$. Describe all the measures with the property that

$$x_j\mu = 0 \text{ in } M(\mathbb{B}^n) \text{ for } j = 1, \ldots, n.$$

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