18.02 Problem Set 9 (due Friday, April 23, 1999)
With Solutions

These problems on multiple integration and surface integrals involve a lot of work. Get started early, even though it's not due until next week. Quite a lot of the work is in Part I.

Part I (20 points)

Hand in the the underlined problems; the others are for practice.

Lecture 27 (Tues. April 13): Triple integrals in rectangular and cylindrical coordinates.

Read: EP, sect. 12.8 to p. 787. Read: SN I, pp. I.2, I.3; Read EP 14.6, pp. 926-929 (concentrate on Exs. 1,2,3); Read EN 14.7, pp. 934-6.

Problems: EP p.791 Probs 5, 9, 11, 15, 23, 25, 27, <u>55</u>, <u>60</u> (give z in terms of r), <u>61</u>. (Solutions in back of book; no. 60: $0 \le r \le R, Hr/R \le z \le H, 0 \le \Theta \le 2\pi$) More problems: I.3/<u>11</u>, <u>12</u>, <u>13</u> (cylindrical coordinates) (I.5). Even more problems EP: 14.6/1, <u>5</u>, 33, <u>39</u> ($\bar{z}$ only; use symmetry, half the region), 43 (S.23); 14.7/<u>9</u>, 12 (vol. only),<u>19</u> (S.24).

Lecture 28 (Thurs. April 15): Triple Integrals in spherical coordinates.

Read: Notes I, p. I.4; EP 14.7, pp. 936-940.

Probs. SN: I.4/<u>14</u>, <u>16</u> (I.5); EP 14.7 p.941 <u>21</u>, <u>26</u>, 29, <u>40</u> (S.24,25).

Lecture 29 (Fri. April 16): Gravitational attraction. Vector fields.

Read: SN sect. G, probs: G.3/<u>3</u>, 4 (S.25). Read: SN, Vector Calculus, section 8. Probs. SN 8.2, nos. <u>1</u>, <u>3</u> (8.2).

Part II (25 points)

Directions: Try each problem alone for 25 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult old problem sets.

1. (Tues. 3 pts: 2,1) (cf. EP 14.6/33 (S.23))

   A rectangular solid has dimensions a, b, c. Its density is 1.

   (a) Find its moment of inertia about an edge of length c. (Place the solid so this edge lies along the z-axis.)

   (b) Suppose the dimensions of the box are 1, 2 and 3. About which edge will the moment of inertia be greatest? smallest? Predict the answer by physical intuition, then verify it by using the formula you found in part (a).

   Solution:

   (a) Moment of inertia about z-axis: $\int_0^a \int_0^b \int_0^c (x^2 + y^2)\, dx\, dy\, dz$.

   Inner: $(x^2 + y^2)z]_0^c = c(x^2 + y^2)$.
   Middle: $c(x^2 y + \frac{1}{3}y^3)]_0^b = c(x^2 b + \frac{1}{3}b^3)$.
   Outer: $c(\frac{1}{3}x^3 b + \frac{1}{3}b^3 x)]_0^a = \frac{abc}{3}(a^2 + b^2)$.

   (Note that this is symmetric in a, b as it should be.)

   (b)

   Should be *biggest* about axis 1 $= \frac{2\cdot 3\cdot 1}{3}(2^2 + 3^2)$
   Should be *smallest* about axis 3 $= \frac{1\cdot 2\cdot 3}{3}(1^2 + 2^2)$
   In between about axis 2 $= \frac{1\cdot 3\cdot 2}{3}(1^2 + 3^2)$

2. (Tues. 3 pts.) Work EP 14.7, p. 940 no. 15 (cf. 14.7/12 (S.24))

   Problem: Find volume of the region D bounded above by the spherical surface $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

   Solution: Certainly D has circular symmetry. Its projection into the xy-plane (its shadow) is a disc of some radius a. To determine a set $x = 0$ and solve

$$z^2 + y^2 = 2 \text{ and } z = y^2 \Rightarrow z^2 + z - 2 = (z-1)(z+2) = 0 \Rightarrow$$
$$z = -2 \text{ (makes no sense) or } z = 1.$$

   So $a = 1$.

   In cylindrical polar coordinates the volume is

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r dz\, dr\, d\theta.$$

Inner:	$rz]_{r^2}^{\sqrt{2-r^2}} = r\sqrt{2-r^2} - r^3$
Middle:	$-\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{4}r^4]_0^1 = -\frac{1}{3} - \frac{1}{4} - \left(\frac{-2^{3/2}}{3}\right)$
Outer:	$2\pi\left(\frac{2\sqrt{2}}{3} - \frac{7}{12}\right)$ (Book: $\frac{\pi}{6}(8\sqrt{2} - 7)$)

3. (Thurs. 2 pts: 1,1)

   Definition: The *average value* of $f(x, y, z)$ over a region D in 3-space is

$$\frac{1}{V(D)} \iiint f(x, y, z)\, dV, \qquad V(D) = \text{volume of } D\,.$$

   Find the average distance of a point in a solid sphere of radius 1 from
   (a) the center of the sphere
   (b) an axis of the sphere
   (c) a point on the surface of the sphere

   Solution:
   (a)

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Inner: $\dfrac{\rho^4}{4} \sin\varphi]_0^1 = \dfrac{\sin\varphi}{4}$, Middle: $\dfrac{-\cos\varphi}{4}]_0^{\pi} = \dfrac{1}{2}$, Outer: $\dfrac{1}{2} \cdot 2\pi = \pi$

   then divide by the volume of the sphere $4/3\pi$ so

$$\text{Answer:} \quad \pi \cdot \frac{3}{4\pi} = 3/4.$$

   (b)

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho \sin\varphi \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Inner: $\dfrac{1}{4}\sin^2\varphi$, Middle: $\dfrac{1}{4}\left(\dfrac{\varphi}{2} - \dfrac{\sin 2\varphi}{4}\right)]_0^{\pi} = \pi/8$, Outer: $2\pi \cdot \pi/8 = \pi^2/4$

So

$$\text{Answer:}\quad \frac{\pi^2}{4}\cdot\frac{3}{4\pi}=\frac{3\pi}{16}.$$

(c) Take the point on the boundary of the solid sphere to be $(0,0,1)$. Then the distance squared from it is

$$x^2+y^2+(z-1)^2=\rho^2-2\rho\cos\phi+1.$$

Thus the integral we wish to evaluate is

$$\int_0^{2\pi}\int_0^1\int_0^\pi(\rho^2-2\rho\cos\varphi+1)^{\frac{1}{2}}\cdot\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

$$\text{Inner:}\quad \frac{1}{3}\rho(\rho^2-2\rho\cos\varphi+1)^{\frac{3}{2}}]_0^\pi=\frac{1}{3}\rho((\rho-1)^3-(\rho+1)^3)$$

$$\text{Middle:}\quad \frac{1}{3}(\frac{2}{5}\rho^5+2\rho^3)]_0^1=\frac{4}{5},\quad \text{Outer:}\frac{8\pi}{5}$$

$$\text{Answer:}\frac{6}{5}.$$

4. (Fri. 3 pts) Prob SN, G.3/1b

Solution: Grav. attractions on vertex of solid icream cone with density ρ

$$=G\int_0^{2\pi}\int_0^{\pi/6}\int_0^a\rho\cdot\frac{\cos\rho}{\rho^2}\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

the integrand being density $\times$ grav. $\times$ dV. Dropping G for the moment

$$\text{Inner:}\frac{\rho^2}{2}\sin\varphi\cos\varphi]_0^a=\frac{a^2}{2}\sin\varphi\cos\varphi$$

$$\text{Middle:}\frac{a^2}{4}\sin^2\varphi]^{\pi/6}=\frac{a^2}{4}\cdot\frac{1}{4}\quad=\frac{a^2}{16}$$

$$\text{Outer:}2\pi\cdot\frac{a^2}{16}=\frac{\pi a^2}{8}$$

$$\text{Answer:}\quad G\cdot\frac{\pi a^2}{8}$$

5. (Fri. 3 pts) Prob SN, G.3/5

Solution: Put the hemisphere with its flat surface up and its pole at the origin. Then do the integral in two pieces: the inner solid cone and the outer shell.

$$=G\int_0^{2\pi}\int_0^{\pi/4}\int^{a/\cos\varphi}\sin\varphi\cos\varphi\,d\rho\,d\varphi\,d\theta$$

(since the top surface is $z=a$, $\therefore\rho\cos\varphi=a$ or $\rho=a/ar\varphi$)

(Drop G during the integration)

Evaluating Cone part:

$$\text{Inner: } \sin\varphi\cos\varphi\cdot\rho]^{a/ar\varphi} = a\sin\varphi,$$

$$\text{Middle: } -aar\varphi]^{2\pi/4} = a\left(1-\frac{\sqrt{2}}{2}\right),$$

$$\text{Outer:} 2\pi\frac{\sqrt{2}}{6}a = \frac{\sqrt{2}\pi a}{3}G$$

So the total is

$$= \pi a\left(2-\sqrt{2}+\frac{\sqrt{2}}{3}\right)G = 2\pi aG\left(1-\frac{\sqrt{2}}{3}\right)$$

6. (Fri. 3 pts)
   (a) Write down the most general vector field, all of whose vectors are perpendicular to the plane $2x - y + 3z = 5$.
   (b) The force of a positive unit dork is radically outward, with magnitude equal to the distance of the dork. Show that the force field in three-space resulting from two positive unit dorks, place on the x-axis at $x = 1$ and $x = -1$, is the same as the force field of a single positive two-unit dork at the origin.
   (c) Write down the magnetic field in 3-space arising from an infinite wire along the y-axis, carrying a unit positive current in the negative direction. (The field is tangent to a circle centered on the y-axis, lying in a plane perpendicular to the y-axis, with direction given by the right hand rule, and magnitude inversely proportional to the distance from the y-axis.)

   Solution:
   (a) If the vectors are perpendicular to $2x - y + 3z = 5$, they are parallel to the normal vector to this plane: $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and so must be of the form

$$\mathbf{F} = f(x, y, z)(2\mathbf{i}) - \mathbf{j} + 3\mathbf{k}$$

   (b) For a unit dork at the origin, $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. For a unit dork at $(-1, 0, 0)$ and $(1, 0, 0)$ therefore the sum of the fields is

$$(x+1)\mathbf{i} + y\mathbf{j} + z\mathbf{k} + (x-1)\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

   which is the same as a two unit dork at the origin.
   (c) Looking at the xz-plane $\mathbf{F} = -\frac{z\mathbf{i}+x\mathbf{k}}{(x^2+z^2)}$.