**18.02 Problem Set 8 (due Thursday, April 8, 1999)**

**Part I:** (10 pts)

Lecture 22. (Thurs. April 1): Differentials, exactness. Criterion for conservative fields; finding potential functions. Read: Notes pp. 2.1–2.5 (p. 2.6 is optional—it is covered in 18.03). EP pp. 980-982 covers about the same material; if you read the book, be sure to read the Notes p. 2.1 for the proof of the theorem, which the book omits. Problems: Work: Notes 2.7 5a (use method 1 of notes), 5b (use method 2), 6ab (do it using both methods) (S. 28)

If $F = \nabla F$, then $f(x, y)$ is called the (mathematical) potential function. The physical potential function is given by $\phi(x, y) = -f(x, y)$, so in physics classes the relation is written $F = -\nabla\phi$.

Lecture 23. (Fri. April 2): Green's Theorem. Read EP, pp. 986-989. Probs: Notes: SP.4/4-C1, a, b, c, SP.5/4-C2, 3, 4, 5 (S.30)

Lecture 24. (Tues. April 6): Two-dimensional flux. Normal form of Green's Theorem. Read: Notes secs. 3 and 4; EP pp. 989–992 is similar, but more condensed; the normal form is called the "vector form" of Green's theorem. Probs: Notes: 3.4/lac,2 (give brief reason or draw picture) 3.4/3, 4, 5 (give a direct calculation; don't use Green) 4.5/4b (S. 31-33)

Lecture 25 (Thurs. April 8): Extensions and applications of Green's Theorem; topology in the plane. Read: Notes: 5 and 6.

Exam 3. (Fri. April 9) 2:05–2:55 in Walker. This will cover double integrals and line integrals through Lecture 24. (Lecture 25 will not be included.)

**Part II**

1. (Tues 4/8.; 5 pts.; 1,2,2)
   (a) Verify that $F = xy\mathbf{i} + x^2\mathbf{j}$ is not a gradient field.
   (b) Try to find a potential function by Method 1 in the notes. What goes wrong?
   (c) Same question for Method 2.
   Solution:
   (a) $\frac{\partial}{\partial y}(xy) = x$, $\frac{\partial}{\partial x}(x^2) = 2x$ so the criterion fails and $\mathbf{F}$ cannot be a gradient.
   (b) The function you get does not have the correct gradient
   (c) Can solve $\frac{\partial}{\partial x} f = xy$, namely $f = \frac{1}{2}x^2y + g(y)$, but then cannot solve $g'(y) = x^2 - \frac{1}{2}x^2 = \frac{1}{2}x^2$ since this is not a function of $y$ alone.
2. (Tues 4/8.; 5 pts.; 2,2,1) The most general two-dimensional field whose direction is radially outward, and whose magnitude depends only on the distance from the origin can be written

$$F = f(r)(x\mathbf{i} + y\mathbf{j})\,.$$

   (a) Show it is conservative. (Use $\partial r/\partial x = x/r$).
   (b) Show it has a potential function of the form $g(r)$.
   (c) Find $g(r)$ if $f(r) = e^{-r}$.
   Solution:
   (a) $\frac{\partial}{\partial y}(xf(r)) = \frac{xy}{r}f'(r) = \frac{\partial}{\partial x}(yf(r))$.

(b) Use integration method – integrate from $(0,0)$ (assuming the vector field is continuous there) along a radius. Then

$$g(r) = \int_0^r \mathbf{F} \cdot \hat{\mathbf{t}} dr = \int_0^r r f(r) dr.$$

(c) If $f(r) = e^{-r}$ then $g(r) = \int_0^r re^{-r}dr = -e^{-r}(r+1)$ by integration by parts.

3. (Tues. 4/8; 5 pts.; 2,3)

(a) Show that the curve $x^{2/3} + y^{2/3} = a^2$ has the parametrization

$$x = a^3 \cos^3\theta\,, \qquad y = a^3 \sin^3\theta\,,$$

and sketch it.

(b) Find the area inside it by using Green's theorem. (The integral can be drastically simplified by using the simplest trigonometric identities. Be careful which form of Green's theorem for area you use, try to get the most symmetric integral.)

(a) For this curve, $x^{2/3} = a^2\cos^2$, $y^{2/3} = a^2\sin^2\theta$ so $x^{2/3} + y^{2/3} = a^2$. Here, $0 \le \theta \le 2\pi$.

(b) By Green's theorem

$$\begin{aligned}
\frac{1}{2}\oint_C -y\,dx + x\,dy &= \text{Area of } R \\
&= \frac{1}{2}\int_0^{2\pi} -(a^3\sin^3\theta)(-a^3\cdot 3\cos^2\theta\cdot\sin\theta) + (a^3+\cos^3\theta)\cdot(a^3\cdot 3\cdot\sin^2\theta\cos\theta)\,d\theta \\
&= \frac{3a^6}{2}\int_0^{2\pi}(\sin^4\theta\cos^2\theta + \cos^4\theta\cdot\sin^2\theta)\,d\theta \\
&= \frac{3a^6}{2}\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta \\
&= \frac{3a^6}{8}\int_0^{2\pi}\sin^2 2\theta\,d\theta \\
&= \frac{3a^6}{16}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta = \frac{3a^6\pi}{8} = \text{Area}
\end{aligned}$$

4. (Tues. 4/8; 3 pts.)

Find the closed curve $C$ in the plane for which $\oint_C x^2y\,dx + x(1-y^2)\,dy$ has the biggest value, and determine this biggest value. (Use Green's theorem, and study the resulting double integral – how would you make it biggest?)

Solution: By Green's theorem

$$\oint x^2ydx + x(1-y^2)dy = \iint_R [(1-^2) - x^2]dA = \iint_R (1-x^2-y^2)$$

The integrand is positive inside the unit circle and negative outside, so takes its maximum value if $C$ is the unit circle, value is $\frac{\pi}{2}$.

5. (Thurs. 4/10; 3 pts.)

Find the flux of the vector field $F = (x^3+y^3)(\mathbf{i}+\mathbf{j})$ through the square of side two with center at the origin and one vertex at $(1,1)$.

Solution: With $S$ this square and $C$ its boundary, $int_C Pdy - Qdx = \iint_S (P_x + Q_y)dA = \int_{-1}^1\int_{-1}^1 3(x^2+y^2)dydx = \int_{-1}^1 (6x^2+2)dx = 8$.

6. (Thurs. 4/10; 4 pts.;2,2)

(a) A function $f(x, y)$ is called *harmonic* if it satisfies Laplace's equation:

$$f_{xx} + f_{yy} = 0$$

Show that every cubic harmonic function $ax^3 + bx^2y + cxy^2 + dy^3$ can be written as a linear combination of two basic cubic harmonic functions $f_1$ and $g_1$, i.e., in the form

$$A(f_1(x, y) + B(g_1(x, y)\,, \qquad A, B \text{ constants}\,.$$

(b) Prove that if $f(x, y)$ is harmonic, and $F = \nabla f$, then the flux of $F$ through any closed curve is zero.

Solution:

(a)

$$\text{For } f = ax^3 + bx^2y + cxy^2 + dy^3\,,$$
$$f_{xx} + f_{yy} = 6ax + 2by + 2cx + 6dy$$
$$\text{vanishes only if } b = -3a\,, c = -3d$$
$$.\text{Thus if } f_1 = x^3 - 3x^2y \text{ and } f_2 = y^3 - 3xy^2$$
$$f(x, y) = af_1 + df_2\,.$$

(b) By Green's Theorem

$$\text{Flux } = \oint_C \vec{F} \cdot \vec{n}\, ds = \int\int_R \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA = \int\int_R (f_{xx} + f_{yy})\, dA = 0\,.$$

These one should be saved for next week, they were not supposed to be on this problem set.

7. (Fri 4/11.; 3 pts.; 2,1)

A regular solid has dimensions $a, b, c$. Its density is 1.

(a) Find its moment of inertia about an edge of length $c$. (Place the solid so this edge lies along the $z$-axis.)

(b) Suppose the dimensions if the box are $1, 2$ and $3$. About which edge will the moment of inertia be greatest? Smallest? Predict the answer by physical intuition, then verify it by using the formula you found in part (a).

8. (Tues 4/15.; 5 pts.; 1,1,3)

Find the average distance of a point in a solid sphere of radius 1 from

(a) the center of the sphere

(b) an axis of the sphere

(c) a point on the surface of the sphere

**Definition:** *The average value* of $f(x, y, z)$ over region $D$ in 3-space is

$$\frac{1}{V(D)} \int\int\int f(x, y, z)\, d\mathcal{V}\,, \qquad V(D) = \text{ volume of } D$$

Use the above definition of average distance. Part (a) is very easy, (b) is almost as easy, (c) is harder.

For (c), you have to choose whether to make the limits easy, but the integrand hard, or to place the sphere so the integrand is easy, but the limits harder. Do the latter: place the sphere so the point is at the origin and the axis along the $z$-axis; this makes the integrand easy and the main issue is the equation of the sphere.