18.02 Problem Set 7 (due Friday April 2, 1:45PM, 2-106)

Part I:

Hand in the underlined problems; the others are for practice.

Lecture 19. Thurs. (March 18): Changing variables in a double integral. Read SN sect CV, supplementing with Examples 1–4 in EP sect 14.9.

Problems: EP page 955 no. $\underline{9}$, page 956 no. $\underline{13}$, SN sect. Vect Calc, page 2.7 $\underline{1}$, Special Problem: The original problem was to evaluate

$$\iint_R \frac{(2x-3y)^2}{(x+y)^2}\,dx\,dy \text{ over } R:$$

where R is the region to the right of the y-axis, below the x-axis and above the line $2x-3y=4$. However this integral is undefined as a Riemann integral (because the integrand is unbounded) or, more sensibly is $+\infty$ as a Lebesgue integral (or as an improper Riemann integral). In short I should have been more careful when I copied this problem, it is bad, bad bad! The difficult is that the line $x+y=0$ where the integrand is infinite is inside the domain R. If instead I had asked you to evaluate

$$\iint_R \frac{(2x-3y)^2}{(x+y+3)^2}\,dx\,dy$$

for the same R we would have had less trouble!

Solutions: EP page 955, no. 9: For $u=xy$, $v=xy^3$ the Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix} = 2xy^3.$$

Thus $\frac{\partial(x,y)}{\partial(u,v)} = 1/2v$. The area integral is therefore

$$\int_3^6 \int_2^4 \frac{du\,dv}{2v} = \int_3^6 \frac{dv}{2v} = \ln 2.$$

EP page 956, no. 13: Since $x = 3r\cos\theta$ and $y = 2r\sin\theta$, the Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix} = 6r.$$

The integral for the volume is

$$\int\int_R (x^2+y^2)6r dr\,d\theta = \int_0^{2\pi}\int_0^1 6r^3(4+5\cos^2\theta)dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3}{2}(4+5\cos^2\theta)d\theta = \frac{39\pi}{2}.$$

Special Problem: Make the change of variables $v = 2x-3y$, $u = x+y$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5.$$

In terms of u and v the region of integration is between $v=0$ and $v=4$ and when $v=2x-3y$ is fixed, u runs from $x=0$, where it is $y=-v/3$, to $y=0$ where it is $x=v/2$. I leave it as an excercise to evaluate the resulting iterated integral

$$\int_0^4 \int_{-v/3}^{v/2} \frac{v^2}{(u+3)^2}\frac{du\,dv}{5}.$$

Lecture 20. Fri. (March 19): Vector fields. Line integrals in the plane.

Read SN sect. on Vect. Calc. section 1 (covered also in EP 15.1, pp. 960-1). Probs. SN Vect. Calc. Exercises 1.1, nos. 3i, ii, iii, 4 (solns p. 1.5). Read EP 15.2 pp. 969-74. Probs. EP page 976, 6, 9, 33a, b, 34, 35, 36 (S.26).

Lecture 21. Tues. (March 29): Path independence; conservative fields in the plane. Read 15.3 to p. 979. Problem: Work SN Vect. Calc. page 2.7 1, 2, 3 (S.27) However, the solution to Problem 3 on page S.27 is incorrect. Rather, $\mathbf{F} = \nabla f = \cos x \cos y\mathbf{i} - \sin x \sin y\mathbf{j}$. The rest is correct.

Remark: The Fundamental Theorem of Calculus for line integrals is Theorem 1. You should be able to state and prove the theorem (in the plane; ignore z). The book writes $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, in the lectures and notes we use $\int_C \mathbf{F} \cdot \mathbf{dr}$. Both have the same meaning: the line integral which calculates the work done *by the field* $\mathbf{F}$ carrying a unit test object along the curve C.

Part II: (15 pt)

Directions: Try each problem alone for 15 minutes. If you subsequently collaborate, this should be acknowledged and solutions must be written up independently.

Problem 1. (Thurs. 5 pt) Work EP page 958 (Misc. Problems) no. 52.

Solution: The integral is

$$I = \int\int_R \exp\left(\frac{x-y}{x+y}\right) dx\, dy.$$

For the suggested change of variables, $u = x - y$, $v = x + y$ the Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Thus $dx\, dy = \frac{1}{2} du\, dv$ The region of integration, R, is the set where $x, y \geq 0$ and $x + y \leq 1$. In terms of the new variables this is $0 \leq v \leq 1$, $u + v \geq 0$, $u - v \geq 0$. Thus

$$I = \frac{1}{2}\int_0^1 \int_{-v}^{v} \exp(u/v) du\, dv = \frac{1}{2}\int_0^1 \left[v \exp(u/v)\right]_{-v}^{v} = \frac{1}{2}\int_0^1 v(e - \frac{1}{e}) = \frac{1}{4}(e - \frac{1}{e}).$$

Problem 2. (Fri. 2 pt) Write down the velocity field for a standard 2-dimensional flow between the lines $x = 0$ and $x = 1$: The flow is upwards, with parabolic cross-section; i.e., along any horizontal line segment between 0 and 1, the velocity vector has magnitude 0 at the two ends, while in between its length increases and decreases so the tips of the vectors lie on a parabola, whose maximum height is 1, in the middle. Indicate reasoning. (This is the way a liquid flows in a pipe if it adheres to the pipe walls.)

Solution: The direction of the vector field is always $\mathbf{j}$. Its magnitude must be $|\mathbf{F}| = 4x(1 - x)$, since it has to be quadratic and vanish at $x = 0$ and $x = 1$ and have size 1 when $x = \frac{1}{2}$. Thus

$$\mathbf{F} = 4x(1 - x)\mathbf{j}.$$

Problem 3. (Fri. 3 pt) Imagine the z-axis represents an infinitely long, uniformly charged wire. The electric force it exerts on a unit charge at the point (x, y) is given by

$$F(x, y) = k(x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)\,.$$

Find by direct calculation of the line integral in each case the work done by the force in moving a unit charge along the paths:

1. line from $(0,1)$ to $(\infty,1)$.
2. circle of radius a with center at origin, traced counterclockwise.
3. line from $(1,0)$ to $(0,1)$ (use integral tables in the book covers).

Solution:

The vector field is

$$\mathbf{F} = \frac{kx\mathbf{i} + ky\mathbf{j}}{x^2+y^2}$$

so the work done along a curve C is

$$k\int_C \frac{xdx+ydy}{x^2+y^2}.$$

1. The curve is x running from 0 to ∞ while $y=1$. Thus the work done is
$$k\int_0^\infty \frac{xdx}{x^2+1} = \frac{k}{2}\ln(x^2+1)\Big]_0^\infty = \infty.$$
So an infinite amount of work must be done (just like 18.02!)
2. The curve is $x = a\cos t$, $y = a\sin t$, t running from 0 to $\pi/2$. Thus $dx = -a\sin tdt$, $dy = a\cos tdt$ and the work done is
$$k\int_0^{\pi/4} \frac{-a^2\cos t\sin t + a^2\sin t\cos t}{a^2}dt = 0.$$
3. Parameterize by $y = 1-x$, $dy = -dx$, x running from 1 to 0, so the work done is
$$k\int_1^0 \frac{xdx+(1-x)(-dx)}{x^2+(1-x)^2} = \int_1^0 \frac{(2x-1)dx}{2x^2-2x+1} = \frac{1}{2}\ln(2x^2-2x+1)\Big]_1^0 = 0.$$

Problem 4. (Fri. 8 pt) Answer the same questions as in SN Vect. Calc. page 2.7 no. 1 for the function $f(x,y) = xy(x+y)$, and the path C given by the quarter circle running from $(0,1)$ to $(-1,0)$.

Solution:

a) Since $f = x^2y + xy^2$, $\mathbf{F} = (2xy+y^2)\mathbf{i} + (x^2+2xy)\mathbf{j}$.

b) Directly: Parameterize the curve by $x = \cos t$, $y = \sin t$, $\pi/2 \le t \le \pi$. Thus $dx = -\sin tdt$, $dy = \cos tdt$ so the line integral is
$$\int_{\pi/2}^{\pi} (-\sin^3 t - 2\cos t\sin^2 t + \cos^3 t + 2\sin t\cos^2 t)dt.$$
Using the formulæ $\sin^3 t = \sin t(1-\cos^2 t)$ and $\cos^3 t = \cos t(1-\sin^2 t)$ the integral becomes
$$\int_{\pi/2}^{\pi} (-\sin t - 3\cos t\sin^2 t + \cos t + 3\sin t\cos^2 t)dt = \cos t - \sin^3 t - \sin t - \cos^3 t\Big]_{\pi/2}^{\pi} = 0.$$
A simpler path between these two point is down the y-axis to the origin and then along the x-axis. The integrand vanishes on both pieces so the integral is zero. By the fundamental theorem
$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(x,y)\Big]_{(-1,0)}^{(0,1)} = 0.$$