18.02 Problem Set 4 (due Friday 2/26, at 1:45 in 2-106)

Part I

Hand in the underlined problems only; the others are for more practice. (After each set of problems is given in the Solution page where they are solved.)

Lecture 7. **Thurs. Feb 18** Functions, partial derivatives, tangent plane:

Read EP 13.2, 13.4, SN TA-1,2. Probls EP p.805-807, nos. 7, 15; 24, 25, 53–58 p. 822-3 nos. 4, 6, 12, 21, 39, 40, 57 solns SN (S.9)

Lecture 8. **Tues. Feb 23** Maximum–minimum problems. Least squares approximation.

Read EP 13.5, SN LS. probs EP p. 834 nos 32, 40, 46 solns SN (S.11) prob. LS-3 no. 1

Part II

Directions: Try each problem alone separately for 20 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous years or to copy directly from anyone else.

Problem 1. (Thurs. 4 pts.) Let $Ax = k$ be a system of linear equations, where A is a nonsingular matrix:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \text{ and } x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let $D = \det(A)$. Since $D \neq 0$, the system has a unique solution $[x_0, y_0, z_0]$, which depends on the matrix A and the vector k. Thus x_0 is a function of the 12 variables $a_1, \ldots, k_3$. Calculate $\partial x_0/\partial k_1$ and $\partial x_0/\partial a_3$, expressing your answer compactly in terms of D, x_0, and as few of the 12 variables as possible.

(*Hint:* Use Cramer's rule and the Laplace expansion by cofactors.)

Solution: By Cramer's rule, x_0 is the quotient of two determinants

$$x_0 = \frac{1}{D} \begin{vmatrix} k_1 & a_2 & a_3 \\ k_2 & b_2 & b_3 \\ k_3 & c_2 & c_3 \end{vmatrix}, \quad D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

We may expand the top determinant along the first row and conclude that it is a linear function of k_1. Thus

$$\frac{\partial x_0}{\partial k_1} = \frac{1}{D} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

Both top and bottom determinants depend on a_3 so we need to expand both along the top row and use the chain rule to see that

$$\frac{\partial x_0}{\partial a_3} = -\frac{x_0}{D} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} + \frac{1}{D}\frac{\partial x_0}{\partial k_1} = \frac{1}{D} \begin{vmatrix} k_2 & b_2 \\ k_3 & c_2 \end{vmatrix}.$$

Expanding out the 2x2 determinants to get an explicit formula is a good idea, but we take no marks off if it is not done.

Problem 2. **(Thurs. 5 pts; 2, 3) The surface of $z = y^2 - x^2$, though curved, can be thought of of as made up entirely of lines.**

1. **Find the equation of the tangent place at the point $P_0 : (x_0, y_0, z_0)$ on the surface.**

2. **Show that if $x = x_0 + t$, $y = y_0 + at$, $z = z_0 + bt$ is a general line through P_0, it is possible to choose a and b in two different ways so that the two corresponding lines lie both in the tangent plane and in the surface.**

Solution:

1. **The surface is $z - y^2 + x^2 = 0$ so its tangent plane at (x_0, y_0, z_0) is** $2x_0(x - x_0) - 2y_0(y - y_0) + (z - z_0) = 0.$
2. **To lie in the tangent plane we must have $2x_0 - 2y_0a + b = 0$. To lie on the surface we must have $z_0 + bt = (y_0 + at)^t - (x_0 + t)^2$. Expanding out this means $z_0 - y_0^2 - x_0^2 + t(b - 2ay_0 - 2x_0) + t^2(a^2 - 1) = 0$ for all t. So all three coefficients must vanish. Now, $z_0 - y_0^2 - x_0^2$ since the point is on the surface. The vanishing of $(b - 2ay_0 - 2x_0)$ means the line is tangent to the surface (so in the tangent plane) and determines b from a, x_0 and y_0. The third condition $a^2 = 1$ has two solutions $a = \pm 1$. So there are two lines of this form in the surface through each point, namely** $x = x_0 + t,\ y = y_0 \pm t\ z = z_0 + (2ay_0 + 2x_0)t.$

Problem 3. (Thurs. 2 pts)

Show $f_{xy} = f_{yx}$ if $f(x, y) = \tan^{-1} \frac{x}{y}$.

Problem 4. (Thurs. 3 pts)

Laplace's equation is $u_{xx} + u_{yy} = 0$.

1. **Verify that $u = \ell n(x^2 + y^2)$ satisfies the equation.**
2. **Find all functions $ax^2 + bxy + cy^2$ (a, b, c constants) that satisfy Laplace's equation, and show they can all be written as $u = Au_1 + Bu_2$, where A and B are arbitrary constants, and u_1 and u_2 are particular functions of the given form.**
3. **Answer the same question as in (b) for $ax^3 + bx^2y + cxy^2 + dy^3$.**

1. **Verify that $u = \ell n(x^2 + y^2)$ satisfies the equation.**
2. **xy satisfies the equation as does $x^2 - y^2$ but x^2 does not. It follows (by directly differentiating) that the solutions are** $Axy + B(x^2 - y^2)$.
3. **Answer the same question as in (b) for $ax^3 + bx^2y + cxy^2 + dy^3$. Differentiating direclyt give $6ax + 6dy + 2by + 2cx$ so that the ones that satisfy Laplace's equation must have $c = -3a$ and $b = -3d$, they are** $A(x^3 - 3xy^2) + B(y_3 - 3x^2y)$.