18.02, Problem set II. Solutions to Part II

Problem 1: (2 pts, after Feb 2)

Derive the formula $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$ for $\alpha>\beta$ between 0 and $\pi/2$ by interpreting the right-hand side as the scalar product of two unit vectors.

Solution: The vector $\mathbf{i}(\alpha)=\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath}$ has unit length and makes an angle α with the x-axis, that is with $\hat{\imath}$. Thus the angle between $\mathbf{i}(\alpha)$ and $\mathbf{i}(\beta)$ is $\alpha-\beta$, assuming as we do that $\alpha>\beta$. The two dot product formulæ give

$$\mathbf{i}(\alpha)\cdot\mathbf{i}(\beta)=|\mathbf{i}(\alpha)||\mathbf{i}(\beta)|\cos(\alpha-\beta)=\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta).$$

Problem 2: (3=1+2 pts, after Feb 2)

1. Show that the four points $(-1,-1,-1)$, $(1,1,-1)$, $(-1,1,1)$ and $(1,-1,1)$ form the vertices of regular tetrahedron (state your reasoning precisely – a regular tetrahedron is one with all sides of equal length).
2. Find the angle in radians between two sides at any one vertex (use a calculator or computer) to two decimal places.

Solution: The length of each side is $2\sqrt{2}$ by computation. At $(-1,-1,-1)$ two of the sides are $\langle 2,2,0\rangle$ and $\langle 0,2,2\rangle$. Their dot product is 4 so if θ is the angle between them them $8\cos\theta=4$, $\cos\theta=1/2$. Thus the angle is $\pi/3=1.05\ldots$.

Problem 3: (3=1+1+2+2 pts, after Feb 4)

Given the vectors $\mathbf{A}=\mathbf{6\hat{i}-2\hat{j}+3\hat{k}}$ and $\mathbf{B}=\mathbf{3\hat{i}+6\hat{j}-2\hat{k}}$

1. Show that $\mathbf{A}$ and $\mathbf{B}$ are perpendicular (orthogonal).
2. Find unit vectors $\hat{\imath}'$ and $\hat{\jmath}'$ in the directions of $\mathbf{A}$ and $\mathbf{B}$.
3. Using the cross product find a third *unit* vector $\hat{k}'$ which is orthogonal to both $\hat{\imath}'$ and $\hat{\jmath}'$ and such that $\hat{\imath}',\hat{\jmath}',\hat{k}$ is a right-handed coordinate system.
4. If $\mathbf{C}=2\hat{\imath}-\hat{\jmath}+\hat{k}$ find constants a,b,c such that $\mathbf{C}=a\hat{\imath}'+b\hat{\jmath}'+c\hat{k}'$ (hint – use the dot product).

Solution:

1. $\mathbf{A}\cdot\mathbf{B}=\mathbf{18-12-6=0}$ so the vectors are orthogonal.
2. $|\mathbf{A}|^{\mathbf{2}}=\mathbf{36+4+9=49}$ so $\hat{\imath}'=\frac{6}{7}\hat{\imath}-\frac{2}{7}\hat{\jmath}+\frac{3}{7}\hat{k}$ and $|\mathbf{B}|^{\mathbf{2}}=\mathbf{9+36+4=49}$ so $\hat{\jmath}'=\frac{3}{7}\hat{\imath}+\frac{6}{7}\hat{\jmath}-\frac{2}{7}\hat{k}$.
3. $\hat{k}'=\hat{\imath}'\times\hat{\jmath}'=-\frac{14}{49}\hat{\imath}+\frac{21}{49}\hat{\jmath}+\frac{42}{49}\hat{k}=-\frac{2}{7}\hat{\imath}+\frac{3}{7}\hat{\jmath}+\frac{6}{7}\hat{k}$.
4. $a=\mathbf{C}\cdot\hat{\imath}'=\frac{15}{7}$, $b=\mathbf{C}\cdot\hat{\jmath}'=-\frac{2}{7}$, $c=\frac{1}{7}$.

Problem 4: (3 pts, after Feb 4)

Consider a tetrahedron with one vertex at the origin and each of the other three vertices on the three coordinate axes, say at the points $\langle a,0,0\rangle$, $\langle 0,b,0\rangle$ and $\langle 0,0,c\rangle$. Let D be the *area* of the side which is not in any one of the coordinate planes and let A, B and C be the areas of the other three sides (which are all right-angled triangles). Show that

$$D^2=A^2+B^2+C^2$$

which is a three-dimensional form of Pythagoras's theorem.

Solution: Area of the non-coordinate plane face can be computed as half the length of the cross product of two of its sides, $\langle -a, 0, c\rangle\times$ and $\langle -a, b, 0\rangle$. Since $\langle -a, 0, c\rangle \times \times\langle -a, b, 0\rangle = \langle -bc, -ac, -ab\rangle$ the square of the area is $D^2 = \frac{1}{4}(b^2c^2 + a^2c^2 + a^2b^2)$. The three coordinate-plane sides have areas $\frac{1}{2}ab$, $\frac{1}{2}ac$ and $\frac{1}{2}bc$ giving $D^2 = A^2 + B^2 + C^2$.

Problem 5: (3 pts, after Feb 5)

A certain wafer manufactoring company makes three colored products by adding dyes to a sugar base (yuck). The dyes are Red (R), Blue (B) and Yellow (Y). On one day there are three production runs, with the three varieties using the following quantities, in ounces, of each dye per 100 pounds:

Type1:R=2,B=7,Y=2
Type2:R=6,B=1,Y=2
Type3:R=0,B=5,Y=3

What would you call the products? No, seriously, the total amounts of each dye used that day (when the Feds came to investigate) was 238 ounces of Red, 98 ounces of Blue and 162 ounces of Yellow. How much of Type3 wafer was produced that day?

Solution: Method 1: Write the system in matrix form as $D = AW$ where D is the vector of dyes used – with entries R, B, Y, W is the vector of amounts (in 100's of pounds) of the three wafer types produces and A is the 3x3 matrix

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 1 & 5 \\ 2 & 2 & 3 \end{bmatrix}$$

Make sure you understand this part! The inverse of this matrix is computed as usual

$$\text{Minor matrix} \begin{bmatrix} -7 & 11 & 12 \\ 18 & 6 & -8 \\ 30 & 10 & -40 \end{bmatrix}$$

$$\text{Cofactor matrix} \begin{bmatrix} -7 & -11 & 12 \\ -18 & 6 & 8 \\ 30 & -10 & -40 \end{bmatrix}$$

$$\text{Transpose cofactor matrix} \begin{bmatrix} -7 & -18 & 30 \\ -11 & 6 & -10 \\ 12 & 8 & -40 \end{bmatrix}$$

The determinant is $6 + 0 + 60 - 0 - 126 - 20 = -80$ so

$$A^{-1} = \begin{bmatrix} \frac{7}{80} & \frac{9}{40} & -\frac{3}{8} \\ \frac{11}{80} & -\frac{3}{40} & \frac{1}{8} \\ -\frac{3}{20} & -\frac{1}{10} & \frac{1}{2} \end{bmatrix}$$

The total of type 3 wafer produced, in 100's of pounds, can be computed from $X = A^{-1}Y$, the third row of which gives $-3 \times 238/20 - 98/10 + 162/2$. Thus $3,550$ pounds of type 3 was made.

Method 2: Use Cramer's rule instead.