18.02 Problem Set 11 (due Friday, May 7, 1999)

Part I (7 points)

Hand in the the underlined problems; the others are for practice.

Lecture 33 (Tues. May 4): Line integrals in space, *curl* **F**, exactness, potentials.

Read SN: Sections 11, 12 and p. 15.1; Problems: SN p. 11.5 nos. 1, 2, 4, 5 (S.40) SN p. 12.4 and 12.5 nos 1, 2, 3ab(ii) (both methods), 5 (S. 41-2)

Lecture 34 (Thurs. May 6): Stokes' Theorem.

Read: SN Section 13, EP Section 15.7. Problems: SP.7 nos. 5-B3, B4, B5, B7 (S. 43-4).

Lecture 35 (Fri. May 7): Stokes' Theorem, cont'd. Applications.

Part II (11 points)

Directions: Try each problem alone for 25 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult old problem sets.

1. (Tues. 2 pts)

   (a) For what value(s) of the constant a, b, c will the field

   $$\mathbf{F} = axyz\mathbf{i} + (bx^2z + z^2)\mathbf{j} + (x^2y + cyz + 2z)\mathbf{k}$$

   be conservative?

   (b) Using these values of the constants, find a potential function for the field, by either method described in the Notes. (Show systematic work.)

   Solution:

   (a) $\mathbf{F}$ will be conservative if $\nabla \times \mathbf{F} = 0$. Since

$$\nabla \times (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$
$$= (x^2 + cz - bx^2 - 2z)\mathbf{i}(axy - 2xy)\mathbf{j} + (2bxz - axz)\mathbf{k}$$

   this vanishes (identically) exactly when $a = 2$, $b = 1$ and $c = 2$.

   (b) Method 1: Calculate the work integral of $\mathbf{F}$ along the curve which starts at the orgin, goes along the x-axis to $(x, 0, 0)$, then in the direction of y to $(x, y, 0)$ then in the z-direction to (x, y, z).

$$f(x, y, x) = \int_0^x 0dx + \int_0^y 0dy + \int_0^z (x^2y + 2yz + 2z)dz = x^2yz + yz^2 + z^2.$$

   Method 2: Solve $f_x = 2xyz$, giving $f = x^2yz + g(y, z)$. Then substitute in $f_y = x^2z + z^2$ giving $g_y = z^2$ so $g = z^2y + h(z)$ and $f = x^2yz + z^2y + h(z)$. Finally substitute this in $f_z = x^2y + 2yz + 2z$ giving $h'(z) = 2z$, so $h(z) = z^2 + C$ and the general answer is $f = x^2yz + z^2y + z^2 + C$ as before.

2. (Thurs. 2 pts) Suppose that in 3-space, $\mathbf{F} = \mathit{curl}\,\mathbf{G}$, where the components of $\mathbf{G}$ have continuous second partial derivatives. Prove that, if S is a closed surface, then

   $$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = 0$$

   in two ways:

(a) by the divergence theorem;
(b) by drawing a closed curve C dividing S into two parts and applying Stokes' theorem to each.

Solution:

(a) By the divergence theorem,

$$\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_D div\,\mathbf{F}dV.$$

Since $\mathbf{F} = curl\,\mathbf{G}$,

$$\begin{aligned} div\,\mathbf{F} = div(curl\,\mathbf{G}) &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z \\ &= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0 \end{aligned}$$

by the equality of mixed second derivatives.

(b) Choose a closed curve C which divides S into two parts (such as a little circular curve). Let S^+ be the half with outward normal which has the correct orientation for Stokes' theorem. Then the other half, S^- with outward normal has the correct orientation for C' which is C run backwards. Applying Stokes' theorem twice

$$\oint_C \mathbf{G}\cdot d\mathbf{r} = \iint_{S^+} curl\,\mathbf{G}\cdot d\mathbf{S},$$
$$\oint_{C'} \mathbf{G}\cdot d\mathbf{r} = \iint_{S^-} curl\,\mathbf{G}\cdot d\mathbf{S}$$

with both S^+ and S^- having the outward orientation of S. Since

$$\oint_C \mathbf{G}\cdot d\mathbf{r} = -\oint_{C'} \mathbf{G}\cdot d\mathbf{r}$$

it follows that

$$\oiint_S \mathbf{F}\cdot d\mathbf{r} = \iint_{S^+} curl\,\mathbf{G}\cdot d\mathbf{S} + \iint_{S^-} curl\,\mathbf{G}\cdot d\mathbf{S} = 0.$$

3. (Thurs, 3 pts) Which of the following differentials are exact? For each one which is, express it in the form df for a suitable function $f(x,y,z)$.
(a) $x^2\,dx + y^2\,dy + z^2\,dz$
(b) $y^2z\,dx + 2xyz\,dy + xy^2\,dz$
(c) $y(6x^2+z)\,dx + x(2x^2+z)\,dy + xy\,dz$

Solution – in each case compute $P_y - Q_x$, $P_z - R_x$, $Q_z - R_y$.

(a) Exact, it is df for $f = \frac{1}{3}(x^3+y^3+z^3)$.
(b) Exact, it is df for $f = xy^2z$.
(c) Exact, it is df for $f = 2x^3y + xyz$.

4. (Thurs 1 pt) Find $curl\,\mathbf{F}$, if $\mathbf{F} = x^2y\mathbf{i} + yz\mathbf{j} + xyz^2\mathbf{k}$.

Solution: From the formula above,

$$curl\,\mathbf{F} = (xz^2 - y)\mathbf{i} - yz^2\mathbf{j} - x^2\mathbf{k}$$

5. (Thurs, 2 pts) The fields $\mathbf{F}$ below are defined for all x, y, z. For each,
(a) Show that curl $\mathbf{F} = \vec{0}$.
(b) Find a potential function $f(x,y,z)$. Use either method, or inspection.
  (i) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
  (ii) $(2xy+z)\mathbf{i} + x^2\mathbf{j} + x\mathbf{k}$
  (iii) $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Solution –

(a) $P_y = 0 = Q_x$, $P_z = 0 = R_x$, $Q_z = 0 = R_y$, so $curl\,\mathbf{F} = \mathbf{0}$. $\mathbf{F} = grad\,f$, $f = \frac{1}{2}(x^2 + y^2 + z^2)$.

(b) $P_y = 2x = Q_x$, $P_z = 1 = R_x$, $Q_z = 0 = R_y$, so $curl\,\mathbf{F} = \mathbf{0}$. $\mathbf{F} = grad\,f$, $f = x^2y + xz$.

(c) $P_y = z = Q_x$, $P_z = y = R_x$, $Q_z = x = R_y$, so $curl\,\mathbf{F} = \mathbf{0}$. $\mathbf{F} = grad\,f$, $f = xyz$.