18.02 Problem Set 10 (due Thursday, April 29, 1999)

Part I (9 points)

Hand in the the underlined problems; the others are for practice.

Lecture 30 (Thurs. April 22): Surface integrals and flux.
Read: SN, vector calculus section 9.
Problems: SN p.9.7 nos. <u>1</u>, <u>2</u>, <u>3</u>, <u>4</u>, <u>6</u>, <u>8</u> (S. 36,37).
Lecture 31 (Thurs. April 22): Applications of flux, divergence theorem.
Read: EP pp. 995-998, 1000-1001.
Lecture 32 (Tues. April 27): Divergence theorem.
Read: SN, Vector Calculus, section 10; EP pp. 1006-1008.
Problems: SN, p. 10.5 nos. 1a, 2, <u>3</u>, 5; p. 10.6 6, 7<u>i</u>, <u>8</u> (S.38,39).

Part II (20 points)

Directions: Try each problem alone for 25 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult old problem sets.

1. (Thurs. 8 pts: 2, 3, 3) Take the surface bounded below by the right-angled cone $z = (x^2 + y^2)^{1/2}$, and above by the unit sphere centered at the origin. The upper and lower surfaces intersect in a circle.

   Let S be the disc having this circle as its boundary; T the spherical cap forming the upper surface, and U the cone forming the lower surface. Orient all three surfaces so that the normal vector points generally upward (i.e, has a positive k-component).

   Calculate the flux of the vector field $F = z\mathbf{k}$ over each of these three surfaces. In each case, do the calculation directly from the definition of the surface integral for flux, as in the Notes and Part I problems.

   Solution. Flux through S :

$$\begin{aligned}\iint_S \vec{F}\cdot\vec{n}dS &= \iint_S z\,dS \\ &= \iint_{x^2+y^2\le 1/2} dx\,dy/\sqrt{2} = \frac{\pi}{2\sqrt{2}}\,.\end{aligned}$$

Since $z = 1/\sqrt{2}$ on the disk.

Flux through T :

$$\begin{aligned}\iint_T \vec{F}\cdot\vec{n}\,dS &= \iint_T z^2\sin\varphi\,d\varphi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4}\cos^2\varphi\sin\varphi\,d\varphi\,d\theta \\ &= \frac{2\pi}{3} - \frac{\pi}{3\sqrt{2}}\,.\end{aligned}$$

Flux through U : On U

$$z = \sqrt{x^2+y^2} = f(x,y)\,, \hat{\mathbf{n}}\,dS = (\mathbf{k} - \mathbf{i} - \mathbf{j})\ dx\,dy$$

$$\iint_U \vec{F}\cdot d\vec{S} = \iint_U z\,dy\,dx = \int_0^{2\pi}\int_0^{1/\sqrt{2}} r\cdot r\,dr\,d\theta = \frac{\pi}{3\sqrt{2}}.$$

2. (Thurs. 3 pts.) A pattern of heat generation and absorption produces the temperature distribution in space (u is the temperature at the point (x, y, z)):
$$u = e^{-x^2-y^2-z^2}.$$
Find the heat flow across a sphere of radius a centered at the origin (cf. EP p. 1000; call the heat conductivity k).
For what radius a will the heat flow across the sphere be greatest?
Solution: The heat flow across a surface is the flux of the vector field $-k\nabla u$ where u is the temerature distribution. In this case,
$$\nabla u = e^{-x^2-y^2-z^2}(-2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$
Thus the flux across the sphere of radius a (outwards) is
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS$$
$$= (-k)\iint(-2)e^{-x^2-y^2-z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot\frac{1}{a}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})dS$$
$$= 8\pi a^3 k e^{-a^2}.$$
The maximum value for this will occur when the derivative vanishes, that is when $8\pi k a^2 e^{-a^2}(3 - 2a^2) = 0$ so $a = \sqrt{\frac{3}{2}}$.
3. (Fri. 4 pts: 2,2) In problem 1 above, you calculated directly the flux of the vector field $F = z\mathbf{k}$ over three surfaces shown in cross-section at the right. The unit with the spherical cap T, the cone U given by $z = r$, and the disc S bounded by the circle in which T and U intersect. In each case, the normal vector is the one with a positive k-component.
The correct value for the flux over the disc S is $\pi\sqrt{2}/4$. Using this:
(a) Use the divergence theorem (and the formula for the volume of a solid cone) to get the flux over the conical surface U. (Watch for orientations!)
(b) Similarly, use the divergence theorem to get the flux over T. (get the volume of the whole ice cream cone and subtract the cone volume to get the volume of the spherical cap on top.)
Solution:
(a) By the divergence theorem,
$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS + \iint_{U'} \mathbf{F}\cdot\hat{\mathbf{n}}dS = \iiint_C (\nabla\cdot\mathbf{F})dV = \mathrm{Vol}(C) = \frac{\sqrt{2}\pi}{12}$$
where U' is U with the opposite normal and C is the solid cone. Therefore,
$$\iint U\mathbf{F}\cdot\hat{\mathbf{n}}dS = -\iint_{U'} \mathbf{F}\cdot\hat{\mathbf{n}}dS = \frac{\sqrt{2}\pi}{6}.$$
(b) Volume of solid is
$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2 \sin\varphi d\rho\, d\varphi\, d\theta = \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{3}.$$
So the volume of the cap is $\frac{2\pi}{3} - \frac{5\sqrt{2}\pi}{12}$. By the divergence theorem
$$\iint_{S'} \mathbf{F}\cdot\hat{\mathbf{n}}dS + \iint_T \mathbf{F}\cdot\hat{\mathbf{n}}dS = \text{Vol of cap}$$

so

$$\iint_T \mathbf{F} \cdot \hat{\mathbf{n}} dS = \frac{2\pi}{3} - \frac{\sqrt{2}\pi}{6}.$$

4. (Fri. 2 pts.) Let S be a smooth closed surface. Show that the field $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ cannot be tangent to S at every point (x, y, z) of S.

   Solution: By the divergence theorem $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iiint_R \nabla \cdot \mathbf{F} dV$. If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $\nabla \cdot \mathbf{F} = 3$ and the volume integral is $3\text{Vol}(R) \neq 0$. Thus the flux of $\mathbf{F}$ through S cannot be zero, so $\mathbf{F} \cdot \hat{\mathbf{n}} = 0$ at some point on S and therefore $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ cannot be tangent to the surface everywhere.
5. (Fri. 3 pts.) Prove that if $f(x, y, z)$ satisfies Laplace's equation (see Notes P), then the flux of its gradient field $\triangledown f$ across any smooth closed surface is 0.

   Solution: If $\mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ then $\nabla \cdot \mathbf{F} = f_{xx} + f_{yy} + f_{zz} = 0$, precisely the given condition that f satisfy Laplace's equation. The by the divergence theorem the flux of ∇f through any closed surface is $\iint_S \nabla a \cdot \hat{\mathbf{n}} dS = 0$.