18.02 Practice Exam 2 — March, 1997 partial solutions soon

Problem 1 (30 points) All six parts refer to the function

$$f(x,y) = \frac{y}{x}\,.$$

1. Draw five reasonably spaced level curves for $f(x,y)$ in the xy-plane; label each with the corresponding value of $f(x,y)$.
2. Let $w = f(x,y)$. Find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.
3. Find the gradient vector $\overrightarrow{\nabla w}$ at any point (a,b) where it is defined, and show by calculation that it is perpendicular to the level curve of $f(x,y)$ passing through (a,b).
4. Find the directional derivative $\frac{dw}{ds}$ at the point $(2,1)$ in the direction of the vector $3\mathbf{i} + 4\mathbf{j}$.

   If you start at this point and go in this direction, approximately what distance would you travel to decrease the value of w by .02?
5. Find the point P on the graph of $f(x,y)$ lying over the point $(2,1)$ in the xy-plane, and find the tangent plane to the graph at the point P.
6. Give an approximate expression for Δw in terms of Δx and Δy, for values of (x,y) close to $(2,1)$. Use it to answer the two questions below.

   Near $(2,1)$, is the value of w more sensitive to x or to y?

   Near $(2,1)$, if an error of $\pm.01$ is made measuring x and y, what is the possible resulting error in the corresponding value of w?

Partial solution: b)

$$\frac{\partial w}{\partial x} = -\frac{y}{x^2},\ \frac{\partial w}{\partial y} = \frac{1}{x} c)$$

$$\overrightarrow{\nabla w} = -\frac{b}{a^2}\mathbf{i} + \frac{1}{a}\mathbf{j},\ b \neq 0$$

The level curve through (a,b) is $y/x = b/a$ that is $x = at,\ y = bt$ which is a straightline with tangent vector $a\mathbf{i} + b\mathbf{j}$.

$$\overrightarrow{\nabla w} \cdot (a\mathbf{i} + \mathbf{j}) = -\frac{b}{a} + \frac{b}{a} = 0.$$

d) The directional derivative is $dw/ds = \overrightarrow{\nabla w} \cdot \hat{t}$ where $\hat{t}$ is a *unit vector* in give direction. Thus the directional derivative is

$$\frac{dw}{ds} = -\frac{1}{4}\mathbf{i} + \frac{1}{2}\mathbf{j} \cdot \frac{3\mathbf{i} + 4\mathbf{j}}{5} = 1/4.$$

Since $\Delta w \sim \frac{dw}{ds}\Delta s$, should go approximately .08 backwards.

e) Point on graph is $P = (2,1,1/2)$. Tangent plane to $z - y/x = 0$ is

$$(z - \frac{1}{2}) + \frac{1}{4}(x-2) - \frac{1}{2}(y-1) = 0.$$

f)

$$\Delta w \sim -\frac{1}{4}\Delta x + \frac{1}{2}\Delta y$$

is more sensitive to changes in y. Possible error is

$$-\frac{1}{4} \times (-0.01) + \frac{1}{2} \times 0.01 = .008.$$

Problem 2 (30 points) A box is to be constructed from wood pieces of uniform thickness so that the top and two opposite sides use a single thickness of wood, while the two ends and the bottom use a double thickness. The volume is to be 24 cubic feet.

What dimensions for the box will use the least amount of wood?

1. Using z for the height and x, y for the other two dimensions, show the function to be minimized has the form (x and y could be interchanged):

$$w = 3xy + 96/x + 48/y\,.$$

2. Find the x, y and z values which minimize w.
3. If the problem is solved using Lagrange multipliers, what is the value of the multiplier λ corresponding to the minimum? (To solve this, you do not need to write down all the equations required by the method.)

Partial solution: a) The amount of wood is

$$w = xy + 2xy + 4xz + 2yz = 3xy + \frac{96}{y} + \frac{48}{x}$$

from top + bottom + sides + ends and given that $xyz = 24$.

b) $\partial w/\partial x = 3y - 48/x^2 = 0$ and $\partial w/\partial y = 3x - 96/y^2 = 0$. Solving gives $x = 2$, $y = 4$ and $z = 3$.

c) One Lagrange equation is $\partial w/\partial x = 3y + 4x = \lambda yz$ so $\lambda = 2$.

Problem 3 (10 points) Suppose a change from xy-coordinates to uv-coordinates is given by the equations

$$x = u^2 - v^2, \qquad y = 2uv\,.$$

Let $w = f(x, y)$. Then after the change of coordinates, w becomes a function of u and v which we shall denote by $g(u, v)$, that is,

$$w = f(x, y) = f(u^2 - v^2, 2uv) = g(u, v)\,.$$

1. Using the chain rule, express $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ in terms of $u, v, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.
2. If $\overrightarrow{\nabla f} = (3, 1)$ at the point $(0, 2)$ in the xy-plane, what is $\overrightarrow{\nabla g}$ at the point $(1, 1)$ in the uv-plane?

Problem 4 (10 points) Where does the tangent plane to the surface $x^2 + 3y^2 + 4z^2 = 8$ at the point $(1, 1, 1)$ intersect the z-axis?

Partial solution: Tangent plane is $2(x - 1) + 6(y - 1) + 8(z - 1) = 0$ intersects the z-axis at $x = y = 0$ that is $z = 2$.

Problem 5 (10 points) Suppose $w = f(x, y, z)$, where $y^2 = xz$. Express $\left(\frac{\partial w}{\partial x}\right)_y$ in terms of the partial derivatives f_x, f_y, and f_z.

Partial solution: Since $y^2 = xz$, $z = y^2/x$ so $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{y^2}{x^2}$.

$$\left(\frac{\partial w}{\partial x}\right)_y = f_x + f_z \left(\frac{\partial z}{\partial x}\right)_y = f_x + z_z \times (-\frac{y^2}{x^2}).$$

Problem 6 (10 points) Let $(f(x, y)$ denote the height of the point (x, y) above sea level. A hiker is ascending a hill. The motion of the hiker is (as observed in the xy-plane, i.e., on a topographic map of the hill) has these two properties:

1. It is always in the direction of $\overrightarrow{\nabla f}$, i.e., perpendicular to the level curves of $f(x, y)$;
2. Its speed (in the xy-plane is inversely proportional to $\left|\overrightarrow{\nabla f}\right|$.

Show that the hiker is ascending at a constant rate.

(Express the velocity vector in terms of f, then use the chain rule to calculate df/dt.)