18.02 Practice Exam 4 — May , 1999 2:05-2:55

Directions: Suggested time: 3 hours.
No calculators or notes. There are 20 questions; each counts 12.5 points.

1. $P:(0,0,0)$ $Q:(0,1,1)$ $R:(1,0,1)$ are three points in space. Find the angle between the vectors PQ and PR.

2. $P:(1,1,0)$ $Q:(2,1,1)$ and $R:(3,2,-1)$ are three points.

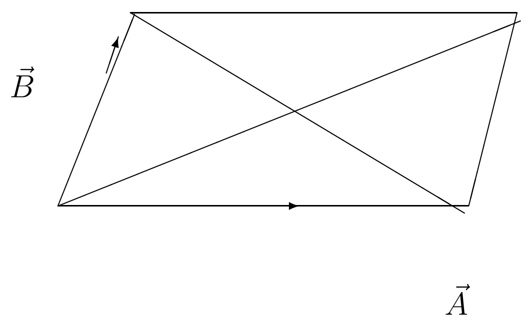
   (a) Find the cross product $PQ \times PR$.

   (b) Using (a), get the equation of the plane through the three points, in the form $ax+by+cz=1$.

3. For what value(s) of the constant c will the system of equations $Ax=0$ have a non-zero solution, if A is the matrix shown and x is the column vector?

$$A = \begin{bmatrix} 1 & 0 & c \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

4. Express in terms of A and B the two diagonals of the parallelogram shown, and then prove using vector algebra that if the two diagonals are perpendicular, the sides of the parallelogram are equal.



5. The position vector of a moving point P is $r = (\cos t)i + (\sin t)j + tk$.

   (a) At what point (a, b, c) does P pass through the surface $z = \pi(x^2 + y^2)$?

   (b) What is its speed at that time?

6. For the function $w = x^2 + xy^2$, and the point $P : (1, 1)$, find

   (a) the value of grad w at P;

   (b) the directional derivative dw/ds at P in the direction of $A = 3i - 4j$;

   (c) starting at P, approximately how far should you go in the direction of A in order to increase the value of w by .01 ?

7. P, V and T for an ideal gas are related by $V = kT/P$, where k is a constant.

   (a) Take $k = 2$, and give an approximate formula telling how ΔV varies with Δp and ΔT when $P = 2$ and $T = 1$.

   (b) Continuing part (a), when $P = 2$ and $T = 1$, is V numerically more sensitive to P or T? (Indicate brief reason.)

8. By using Lagrange multipliers, find the point on the plane $x + 2y + z - 1 = 0$ which is closest to the origin. (Minimize the square of the distance from the origin. If you don't use Lagrange multipliers, do it some other way for 6 points credit; L.M.s are the easiest way.)

9. Let $w = f(x, y)$ and r, θ be the usual polar coordinates. By using the chain rule, find the 2×2 matrix A (such that the entries of A are explicitly given functions)

$$\begin{pmatrix} w_r \\ w_\theta \end{pmatrix} = A \begin{pmatrix} w_x \\ w_y \end{pmatrix}.$$

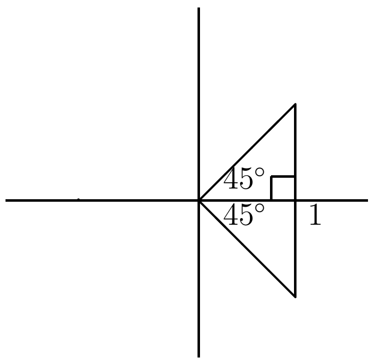
10. For the surface $x^2 - y^2 + 2z^2 = 8$, find the tangent plane at $(1,1,2)$, in the form

$$ax + by + cz = 1\,.$$

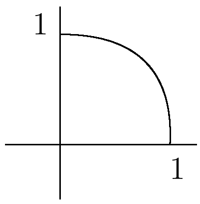
11. Evaluate by changing the order of integration:

$$\int_0^2 \int_{x^2}^4 xe^{-y^2}\,dy\,dx\,.$$

12. Set up an iterated integral in polar coordinates for the moment of inertia about the origin of the triangular plate shown. Take the density $= 1$. Do not evaluate the integral.

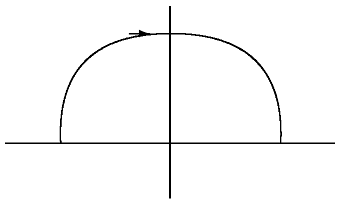


13. Let $F = xy\,i + j$. Find the work done by F going over the quarter-circular path shown, going from $(1,0)$ to $(0,1)$.



14. (a) Express the field $F = x(x-2y)i + (2y - x^2)j$ in the form $F = \nabla f$, for some function $f(x,y)$. Use a systematic method; show work.

    (b) Use this to evaluate $\int_C F \cdot dr$ over the portion of the ellipse given by the graph of $x^2 + 4y^2 = 4$ and lying in the upper half-plane; integrate in the direction left to right.



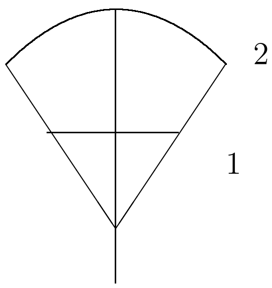
15. How should the constants a and b be related if for any simple closed curve C

$$\oint_C ay\,dx + bx\,dy = \text{area inside } C \quad ?$$

(Give the most general relation; indicate reasoning.)

16. The solid shown has as its sides the vertical right circular cone, with vertex at the origin and 60-degree vertex angle. Its top is a portion of the sphere of the radius 2, and its bottom is horizontal and flat, intersecting the cone at a point having distance 1 from the origin. (Cross-section is picture.)

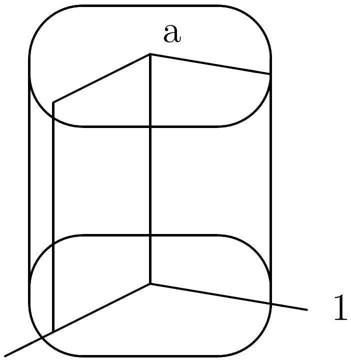
Set up (but do not evaluate) an iterated triple integral in spherical coordinates giving the gravitational attraction of the solid on a unit mass placed at the origin. (Take the density = 1, and the gravitational constant $G = 1$.)



17. Let $F = xi + yj + zk$, and S be the closed cylindrical surface pictured; its sides are the cylinder $x^2 + y^2 = 1$; its top and bottom are horizontal, at heights a and O, respectively.

   (a) Find the flux of F over the top and bottom discs.

   (b) Using part (a) and the divergence theorem, find the flux of F across the side cylinder.



18. Referring to problem 17, find the flux of F across the side cylinder by evaluating a surface integral directly (i.e., without using the divergence theorem).

19. For what value(s) of the constants a and b will the line integral

$$\int_P^Q (axy + yz)\,dx + (x^2 + 2y + xz)\,dy + (y^2 + bxy)\,dz$$

be independent of the path? (Show work.)

20. By using Stokes' theorem, prove that $\oint_C y dx + z dy + x dz = 0$ around any simple closed curve lying in the plane $x - 2y + z = 3$.

Brief Solutions:

1.

$$\vec{PQ} = \hat{\mathbf{j}} + \hat{k}\,, \quad \vec{PR} = \hat{\mathbf{i}} + \hat{k}$$

so

$$\vec{PQ} \cdot \vec{PR} = 1\,, \quad \left|\vec{PQ}\right| = \sqrt{2} = \left|\vec{PR}\right|$$

hence $\cos\gamma = 1/2\,, \gamma =$ angle between $\vec{PQ}$ and $\vec{PR}$. Since angle is less that $90°$, angle $= 30°$.

2.

$$\vec{PQ} = \hat{\mathbf{i}} + \hat{k}\,, \quad \vec{PR} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{k}$$

(a) $\vec{PQ} \times \vec{PR} = \hat{k} + \hat{\mathbf{j}} + 2\hat{\mathbf{j}} - \hat{\mathbf{i}} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{k}$.

(b) Since this is a normal to the plane, it must be

$$-x + 3y + z = d$$

substituting $d = 2$ so plane is

$$-\frac{x}{2} + \frac{3y}{2} + \frac{3}{2} = 1\,.$$

3. It will have a non-zero solution if and only if

$$\det A = 1 + 4c - c + 2 = 0 \qquad \text{i.e.} \quad c = -1\,.$$

4. Diagonals are $A - B$ and $A + B$. These are perpendicular if $(A - B) \cdot (A + B) = 0$, so

$$|A|^2 = |B|^2$$

hence $|A| = |B|$ and the side lengths are equal.

5.

$$\vec{r} = \cos t\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + t\hat{k}$$

(a) Passes through $z = \pi(x^2 + y^2)$ if

$$z = t = \pi \quad \text{i.e. at } (a, b, c) = (-1, 0, \pi)\,.$$

(b)

$$\frac{d\vec{r}}{dt} = -\sin t\hat{\mathbf{i}} + \cos t\hat{\mathbf{j}} + \hat{k} = -\hat{\mathbf{j}} + \hat{k} \text{ at } t = \pi$$

so speed $= \sqrt{2}$.

6. $w = x^2 + xy^2$

(a) $\vec{\nabla} w = (2x + y^2)\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ at $(1, 1)$.

(b)

$$\begin{aligned} \frac{dw}{ds} &= \vec{\nabla} w \cdot \vec{n}\,, \vec{n} = \frac{3i - 4j}{5} \\ &= \frac{1}{5} \text{ in direction of } 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}\,. \end{aligned}$$

(c) Approximately $.01 \times 5 = .05$.

7. (a)

$$\Delta V \cong \frac{\partial V}{\partial T}\Delta T + \frac{\partial V}{\partial P}\Delta P$$

$$\frac{\partial V}{\partial T} = \frac{k}{p} = 1\,, \qquad \frac{\partial V}{\partial p} = -\frac{kT}{p^2} = -\frac{1}{2}$$

$$\text{so } \Delta V \cong \Delta T - \frac{1}{2}\Delta P\,.$$

(b) More sensitive to ΔT.

8. Lagrange multiplier

$$x + 2y + z - 1 - \gamma(x^2 + y^2 + z^2)$$

$$\begin{array}{lll} \partial_x : & 1 - 2x\gamma & = 0 \quad \gamma = \frac{1}{2x} = \frac{1}{y} = \frac{1}{2z} \\ \partial_y : & 2 - 2y\gamma & = 0 \qquad 2x = y = 2z \\ \partial z : & 1 - 2z\gamma & = 0 \quad x + 4x + x - 1 = 0 \end{array}$$

$$x = 1/6\,, y = 1/3\,, z = 1/6\,.$$

Closest point is $(\frac{1}{6}\,, \frac{1}{3}\,, \frac{1}{6})$.

NB It is *easier* to look at the normal $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{k}$ and see where the line in this direction meets the plane

$$x = t\,, y = 2t\,, z = t \quad : \quad t + 4t + t = 1\,, t = 1/6$$

Point is $(\frac{1}{6}\,, \frac{1}{3}\,, \frac{1}{6})$.

9. $w = f(x, y)\,, x = r\cos\theta\,, y = r\sin\theta$

$$\begin{array}{lll} w_r = & \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} & \qquad w_x = \partial f/\partial x \\ w_\theta = & \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} & \qquad w_y = \partial f/\partial y \end{array}$$

$$\begin{pmatrix} w_r \\ w_\theta \end{pmatrix} = A \begin{pmatrix} w_x \\ w_y \end{pmatrix}, A = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$\text{so } A = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}.$$

10. $f = x^2 - y^2 + 2z^2 - 8$. Normal to surface is

$$\vec{\nabla} f = 2x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}} + 4z\hat{k}$$
$$\vec{N} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{k} \text{ at } (1,1,2)\,.$$

Tangent plane is $\vec{N} \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{k} = \hat{N} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{k})$

$$2x - 2y + 8z = 2 - 2 + 16 = 16$$
$$\frac{x}{8} - \frac{y}{8} + \frac{z}{2} = 1\,.$$

11.

$$\int_0^2 \int_{x^2}^4 xe^{-y^2}\,dy\,dx$$
$$= \int_0^4 \int_{x^2}^{y^{1/2}} xe^{-y^2}\,dx\,dy$$
$$= \int_0^4 e^{-y^2} \left[\frac{x^2}{2}\right]_0^{y^{1/2}} dy$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^4 ye^{-y^2}\,dy = \frac{1}{2}\left[-\frac{1}{2}e^{-y^2}\right]_0^4 \\
&= \frac{1}{4}\left(1-e^{-16}\right)\,.
\end{aligned}
$$

12. Twice integral for upper half:

$$
2\times\int_0^{\pi/4}\int_0^{1/\cos\theta} r^2 r\,dr\,d\theta\,.
$$

$$
(x = r\cos\theta = 1)
$$

13. $\vec{F} = xy\hat{\mathbf{i}} + \hat{\mathbf{j}}$. Parameterization

$$
x = \cos t\,, y = \sin t \quad 0 \le t \le \pi/2\,.
$$

$$
\int_c \vec{F}\cdot d\vec{r} = \int_c xy\,dx + dy
$$

$$
\begin{aligned}
&= 1 + \int_0^{\pi/2} \cos t\sin t(-\sin t)\,dt \\
&= 1 - \int_0^{\pi/2} \sin^2 t\frac{d\sin t}{dt}\,dt \\
&= 1 - \int_0^1 s^2\,ds \qquad s = \sin t \\
&= 1 - \frac{1}{3} = 2/3\,.
\end{aligned}
$$

14. (a)

$$
\vec{F} = F_1 b\hat{f}i + F_2\hat{\mathbf{j}}\,, F_1 = x(x-2y)\,, F_2 = 2y - x^2\,.
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= F_1 \text{ so } f = \frac{x^3}{3} - x^2y + g(y) \\
\frac{\partial f}{\partial y} &= -x^2 + g' = 2y - x^2\,, g = y^2
\end{aligned}
$$

Thus $\vec{F}$ is the gradient $\frac{x^3}{3} - x^2y + y^2$.

(b)

$$\int_c \vec{F} \cdot d\vec{r} = f(P_1) - f(P_0) \text{ if } \vec{F} = \vec{\nabla} f \,.$$
$$P_1 = (+2, 0)\,, P_0 = (-2, 0) \text{ so}$$
$$\int_c \vec{F} \cdot d\vec{r} = \frac{16}{3} \,.$$

15. By Green's theorem

$$\oint_C ay\,dx + bx\,dy = \int\int_S (b - a)\,dx\,dy \,.$$

For this to be 1/ the area, must have $b - 1 = 1$ always.

16. Question is confusing without picture. I think it should be:

In spherical coordinate $Gm = 1$)

$$\int_0^{2\pi} \int_0^{\pi/3} \int_{1/2\cos\varphi}^{2} \cos\varphi \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Bottom plane in $z = \frac{1}{2} = \rho\cos\varphi$.

17. (a)

$$\int\int_T \vec{F} \cdot \vec{n} \, dS = \int\int_T z \, dS = a\pi \quad \text{through top}$$

$$\int\int_B \vec{F} \cdot \vec{n} \, dS = 0 \, .$$

(b)

$$\div \vec{F} = 3 \text{ so} \qquad \int\int\int_D \div \vec{F} \, dV = 3\pi a$$

where D is the interior of the cylinder. Thus the flux through the sides is $2\pi a$.

18.

$$\int\int_S \vec{F} \cdot \vec{n} \, dS = \int\int_S (x^2 + y^2) \, dS$$

For the side surface $\vec{n} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\,, x^2 + y^2 = 1$ so the flux through the sides in the area, $2\pi a$.

19. (Note this has been corrected.) The line integral is independent of the path only if

$$\vec{F} = (axy + yz)\hat{\mathbf{i}} + (x^2 + 2yz + xz)\hat{\mathbf{j}} + (y^2 + bxy)\hat{k}$$

is a gradient.

$$\frac{\partial F_1}{\partial y} = ax + z \, , \frac{\partial F_2}{\partial x} = 2x + z \Rightarrow a = 2$$

$$\begin{aligned}\frac{\partial F_1}{\partial z} &= y\,, \frac{\partial F_3}{\partial x} = by \Rightarrow b = 1\\ \frac{\partial F_2}{\partial z} &= 2y + x\,, \frac{\partial F_3}{\partial y} = 2y + bx \Rightarrow b = 1\end{aligned}$$

So only if $a = 2\,, b = 1\,.$

20. By Stokes' theorem

$$\begin{aligned}\oint_C y\,dx + z\,dy + x\,dz = \int\int_S \cos \vec{F} \cdot d\vec{S}\\ \vec{F} = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} + x\hat{k} \text{ so}\\ \text{curl}\vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{k}\\ \partial x & \partial y & \partial z\\ y & z & x \end{vmatrix}\\ = -\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{k}\end{aligned}$$

Since $d\vec{S} = \vec{n}\,d\vec{S}$, $\vec{n} = (\hat{\mathbf{i}} - 2\hat{j} + \hat{k})/\sqrt{6}$

$$\text{curl}\vec{F} \cdot \vec{n} = 0 \text{ so } \int\int_S \text{curl}\vec{F} \cdot d\vec{S} = 0$$

for any surface contained in $x - 2y + z = 3$. Thus

$$\oint_C y\,dx + z\,dy + x\,dz = 0\,.$$