Name________________________ **e-mail**________________________

Rec.Teacher____________________ **Day-hr**____________________

18.02 EXAM 4 — APRIL 30, 1999 2:05-2:55

Attempt all questions. Each problem is worth 20 points, although some are harder than others!

1. Using the Divergence Theorem, evaluate the flux integral

$$\oiint_S \mathbf{F}\cdot\hat{n}\,dS$$

where $\mathbf{F}=x^2\mathbf{i}+(y+z^3)\mathbf{j}+y^2\mathbf{k}$, S is the surface of the sphere $x^2+y^2+z^2=4$ and $\hat{n}$ is the unit outward normal.

Solution: By the divergence theorem this flux integral is equal to the volume integral

$$\iiint_{x^2+y^2+z^2\le 4}\text{div}\mathbf{F}\,dV=\iiint_{x^2+y^2+z^2\le 4}(2x+1)\,dV=\frac{32\pi}{3}$$

by symmetry and the formula for the volume of a sphere.

2. Compute the surface area of the part of the surface $z+x^2+y^2=1$ in $z\ge 0$.

Solution: The surface area is given by the integral $\iint_S dS$. We may use the variables x and y on the surface, the 'shadow region' is $x^2+y^2\le 1$ and the surface measure is

$$dS=(1+f_x^2+f_y^2)^{\frac{1}{2}}\,dx\,dy=(1+4x^2+4y^2)^{\frac{1}{2}}\,dx\,dy,\ f(x,y)=1-x^2-y^2.$$

Thus the surface area is

$$\iint_{\{x^2+y^2\le 1\}}(1+4x^2+4y^2)^{\frac{1}{2}}\,dx\,dy=\int_0^{2\pi}\int_0^1(1+4r^2)^{\frac{1}{2}}r\,dr\,d\theta$$

$$=2\pi\Big[\frac{1}{12}(1+4r^2)^{\frac{3}{2}}\Big]_0^1=\frac{1}{6}\pi(5^{\frac{3}{2}}-1).$$

3. Let D be the three dimensional region between the two surfaces

$$\begin{aligned} z &= (x^2+y^2)^2 \\ z &= 8+2(x^2+y^2)\,. \end{aligned}$$

(a) Sketch the region D.

(b) Using cylindrical (polar) coordinates compute the moment of inertia of D, assumed to have unit density, around the z-axis.

Solution: a) Anything reasonable! It lies over $x^2+y^2\le 4$ with $(x^2+y^2)^2\le z\le 8+2(x^2+y^2)$.

b)

$$\text{MoI}=\int\int\int_D(x^2+y^2)dV=\int_0^{2\pi}\int_0^2\int_{r^4}^{8+2r^2}r^3dzdrd\theta$$

$$=2\pi\int_0^2 r^3(8+2r^2-r^4)dr=2\pi(2\cdot 2^4+2^6/3-2^8/8)=128\pi/3.$$

4. Let D be the region in three dimensional space consisting of the points (x, y, z) with $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x^2 + y^2 + z^2 \leq 4$.

   Assuming where needed that D has unit density, write down integral formulae in terms of spherical coordinates (DO NOT EVALUATE) for

   (a) The average over D of the distance from the z-axis.

   (b) The moment of inertia of D, with unit density, around the z-axis.

   (c) The $\hat{k}$ component of the gravitational force exerted by D on a unit mass at the origin.

   Solution: The region in spherical coordinates is $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq \pi/2$ and $0 \leq \rho \leq 2$. Thus the three integrals are

   (a) $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^3 \sin^2 \phi d\rho d\phi d\theta/(4\pi/3)$ (or an integral for the volume can be written out).

   (b) $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^4 \sin^3 \phi d\rho d\phi d\theta$.

   (c) $G \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \sin \phi \cos \phi d\rho d\phi d\theta$.

5. Let D be 'ice-cream cone' consisting of the points in the unit ball (=solid unit sphere) where the 'azimuth' (angle with the positive direction of the z-axis) $\varphi < \pi/4$. Assuming that it has unit density, compute the gravitational force D exerts on a unit mass at the origin.

   Solution: Place D so that its axis is the z-axis (as indicated). Then in spherical polar coordinates the $\mathbf{k}$ component (and hence the actual) gravitational force on a unit mass at the origin is

$$G \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \frac{\cos \varphi}{\rho^2} \rho^2 \sin \varphi d\rho \, d\varphi \, d\theta = 2\pi G [\frac{1}{2} \sin^2 \varphi]_0^{\pi/4} = \pi G/2.$$