18.02 Practice Exam 4 — for April 25, 1997

Directions: Suggested time: 50 minutes.

1. (20 points) The solid D is represented as the region of space between the graph of $z = 1 - x^2 - y^2$ and the xy-plane. Find its moment of inertia about the z-axis, if the density is ℓ.

   Solution: The moment of inertia around the z-axis is the integral

$$\iiint_D \ell(x^2+y^2)dV = \iint_R \int_0^{1-x^2-y^2} \ell(x^2+y^2)dz\,dx\,dy$$
$$= \iint_R \ell(x^2+y^2)(1-x^2-y^2)dx\,dy$$

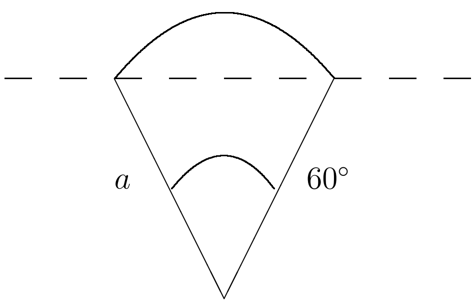
   where the shadow region R is $x^2 + y^2 \leq 1$. Introducing polar coordinates this becomes

$$\text{MoI} = \int_0^{2\pi} \ell \int_0^1 (1-r^2)r^2\, r dr\, d\theta = 2\pi\ell\Big[\frac{1}{4}r^4 - \frac{1}{6}r^6\Big]_0^1 = \frac{\pi}{6}\ell.$$

2. (20 points) A solid D has the shape of a filled ice cream cone: a right circular cone, surmounted by a slice from a sphere of radius a.

   The cone has slant height a and a vertex angle 60 degrees as pictured.

   Take the density to be 1, and find the gravitational attraction of D on a unit mass placed at the vertex. (Set up and evaluate a triple integral in spherical coordinates.)



   Solution: By symmetry, the gravitational attraction on a unit mass at the origin is in the direction of the z-axis and hence equal to its $\mathbf{k}$ component. This is

$$G\iiint_D \frac{\cos\varphi}{\rho^2}dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^a \cos\varphi\sin\varphi d\rho\, d\varphi\, d\theta$$

$$= G2\pi a\Big[\frac{1}{2}\sin^2\varphi\Big]_0^{\pi/6} = Ga\pi/4.$$

3. (5 points) Modify the iterated triple integral of the preceding problem so it now calculates the gravitational attraction at the vertex due to the part of D lying above the top of the cone (i.e., to the spherical slice). Just set up the integral; don't evaluate it.
   Solution: Instead the integral becomes

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{a\cos\pi/6}{\cos\varphi}}^a \cos\varphi\sin\varphi d\rho\, d\varphi\, d\theta.$$

4. (10 points) Suppose the vector field F in the xy-plane satisfies $F = \nabla f$, where

$$f_{xx} + f_{yy} = 0 \quad \text{for all } x, y\,.$$

   Show that F has no net flux through any simple closed curve C.
   This should not really have been on this practice exam. Better do one for flux through a surface.
   Solution: $\mathbf{F} = f_x\mathbf{i} + f_y\mathbf{j}$. The flux form of Green's theorem for a simple closed curve C is

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}ds = \iint_R div\,\mathbf{F}dA.$$

   Since $div\,\mathbf{F} = \frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_y) = f_{xx} + f_{yy} = 0$ by assumption. Thus the flux integral must vanish.
5. (45 points; 5, 15, 25)
   Let S be the hemisphere of radius 1, with base in the xy-plane and center at the origin, and T be its base (the disc of radius 1 and center at the origin in the xy-plane). Both surfaces are oriented so their unit normal vectors point *upwards*, i.e., this is the direction of positive flux. Let $\mathbf{F} = (3z+1)\mathbf{k}$.
   (a) Find the flux of F over T, by inspection.
   (b) Using your answer to part (a), find the flux of F over S *by using the divergence theorem.*
   (c) Find the flux of F over S directly, by calculating a surface integral using spherical coordinates.
   Solution:

(a) On T the normal upward normal is $\mathbf{k}$ so $\mathbf{F}\cdot\mathbf{j}=1$ and hence the flux upward through T is the area π.

(b) If $\hat{\mathbf{n}}$ is the upward unit normal to both S and T then by the divergence theorem

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS - \iint_T \mathbf{F}\cdot\hat{\mathbf{n}}dS = \iint_D \nabla\cdot\mathbf{F}dV.$$

The divergence is $\nabla\cdot\mathbf{F}=3$ so

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = 3\mathrm{Vol}(D)+\pi = 3\pi.$$

(c) The flux integral written explicitly is

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}dS = \int_0^{2\pi}\int_0^{\pi/2}(3z+1)z\sin\varphi d\varphi\, d\theta.$$

Here $z=\cos\varphi$ on S so this flux integral is

$$2\pi\int_0^{\pi/2}(3\cos^2\varphi\sin\varphi+\cos\varphi\sin\varphi)d\varphi$$

$$= 2\pi\Big[-\cos^3\varphi-\frac{1}{2}\cos^2\varphi\Big]_0^{\pi/2} = 3\pi.$$