Name____________________ **e-mail**____________________
Rec.Teacher____________________ **Day-hr**____________________

18.02 Exam 3 — April 9, 1999

1. Compute the volume of the region between the xy plane and the surface $z = -x^2 - y^2 + 4$.

   Solution: The volume is given by the double integral over the region, $R = \{x^2 + y^2 \leq 4\}$

$$\iint_R (4 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^2 (4 - r^2) r \, d\theta \, dr$$

$$= 2\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2 = 8\pi.$$

2. Evaluate the integral

$$\int_0^1 \int_x^1 \cos(y^2) \, dy \, dx \,.$$

   Solution: Change the order of integration and the integral becomes

$$\int_0^1 \int_0^y \cos(y^2) \, dx \, dy = \int_0^1 y \cos(y^2) dy = \left[\frac{1}{2} \sin(y^2) \right]_0^1 = \frac{1}{2} \sin 1.$$

3. Using polar coordinates evaluate the integral

$$\int \int_R \exp(x^2 + y^2) \, dA$$

   where R is the disc of radius 1 and center the origin.

   Solution: In polar coordinates the integral is

$$\int_0^{2\pi} \int_0^1 \exp(r^2) \, r dr \, d\theta = 2\pi \left[\frac{1}{2} \exp(r^2) \right]_0^1 = \pi(e - 1).$$

4. (a) For what values of the constant a is
$$\mathbf{F} = (4x^3 + 2xy + ay^2)\mathbf{i} + (x^2 + 2y)\mathbf{j}$$
a conservative vector field?

   (b) For $a = 0$ compute
$$\int_C \mathbf{F} \cdot d\mathbf{r}$$
where C is the semicircule with radius 1, center $(0,0)$ from $(1,0)$ to $(-1,0)$ in the upper half-plane.

Solution:

(a)
$$\frac{\partial}{\partial y}(4x^3 + 2xy + ay^2) = 2x + 2ay = \frac{\partial}{\partial x}(x^2 + 2y) = 2x$$
exactly when $a = 0$.

(b) Since the vector field is conservative the integral is the same for any contour with these two endpoints, for instance the segment $[-1,1]$ of the x-axis. The integral along the x-axis is $\int_{-1}^{1} 4x^3 dx = 0$.

5. (a) For the vector field $\mathbf{F} = y(x^2 + y^2)^2\mathbf{i} - x(x^2 + y^2)^2\mathbf{j}$ compute $\text{div}(\mathbf{F})$

   (b) For this vector field and the curve C which is the circle of radius 2 with center the origin and positive orientation, use Green's theorem to compute the flux integral
$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds\,.$$

Solution:

(a)
$$\text{div}(\mathbf{F}) = \frac{\partial}{\partial x}[y(x^2 + y^2)^2] - \frac{\partial}{\partial y}[x(x^2 + y^2)^2]$$
$$= 4xy(x^2 + y^2) - 4xy(x^2 + y^2) = 0.$$

(b) 0

6. Sketch the parameterized curve

$$x = \cos t\,,\quad y = \sin^3 t\,, 0 \le t \le 2\pi\,.$$

(a) Why is it closed?
(b) Why is it simple?
(c) Use Green's theorem to express the area inside the curve as a single integral. Do not evaluate it.

OR simply:

Use Green's theorem to express the area inside the simple closed curve

$$x = \cos t\,,\quad y = \sin^3 t\,, 0 \le t \le 2\pi,$$

as a single integral. Do not evaluate it.

Solution: [There is of course a small conceptual problem here that the curve is not smooth, but I don't suppose that anyone will notice.]

(a) $x(2\pi) = x(0)$, $y(2\pi) = y(0)$.
(b) $\sin t$ takes each value between -1 and 1 twice for t in the interval $[0, 2\pi]$ but at these points $\sin t$, and hence $\sin^3 t$ has opposite sign.
(c)

$$\oint_C -y\,dx = \int_0^{2\pi} \sin^4 t\,dt.$$

7. Let R be the region in the first quadrant ($x \ge 0, y \ge 0$) where $3x + y \le 1$. Using the change of variable formula for double integrals rewrite the integral

$$\int_R\int (3x+y)^4 \exp(x+y)\,dA$$

as an iterated integral in terms of the new variables $u = 3x + y$ and $v = x + y$. Give limits of integration but do not evaluate.

Solution: For $u = 3x + y$ and $v = x + y$ the Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2,$$

so $\partial(x,y)/\partial(u,v) = 1/2$. On $x = 0$, $v = u$ and on $y = 0$, $v = u/3$ so the integral becomes

$$\frac{1}{2}\int_0^1 \int_{u/3}^{u} u^4 \exp(v)\,dv\,du.$$