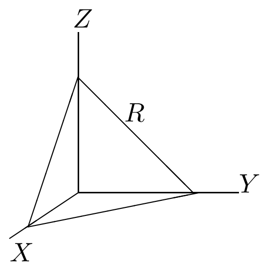
18.02 Practice Exam 3 — April, 1999

Directions: Suggested time: 70 minutes.

Problem 1 (15 points) The solid R is the piece of the first octant cut off by the plane

$$x + y + z = 1$$

Set up an iterated double integral in rectangular coordinates which gives the volume of R. (Give the integrand and limits of integration, but *do not evaluate*.)



Problem 2 (15 points) A circular disc has radius a and A is a point on its circumference. The density at any point P on the disc is equal to the distance of P from A. Set up an iterated double integral in polar coordinates which gives the mass of the disc. Place A at the origin. (Give the integrand and limits, but to not evaluate the integral.)

Problem 3 (15 points) Evaluate the integral $\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$ by changing the order of integration. (Sketch the region of integration first.)

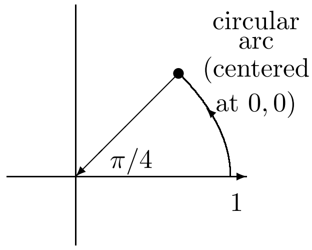
Problem 4 (10 points) Change $\int_0^1 \int_{-x}^{x} (x^2+y^2)^{3/2}\, dy\, dx$ to an interated integral in polar coordinates. (Do not evaluate it.)

Problem 5 (30 points; 10 each)

a) $\mathbf{F} = (axy+y^2)\mathbf{i}+(x^2+bxy+1)\mathbf{j}$; a, b are constants. Show that F is conservative $\iff a = 2,\ b = 2$.
b) Taking $a = 2$, $b = 2$, find $f(x, y)$ so that $\mathbf{F} = \nabla f$.
c) Still taking $a = 2$, $b = 2$, show $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any curve C beginning and ending on the x-axis.

Problem 6 (30 points; 15, 5, 10)

a) Evaluate $\oint_C -x^2y\, dx + xy^2\, dy$ by Green's theorem, if C is the closed curve as pictured passing through $(1, 0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(0, 0)$, and back to $(1, 0)$.



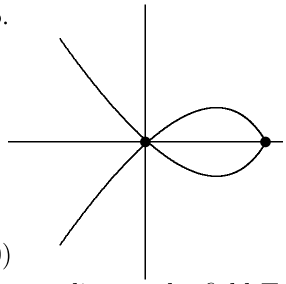
b) Show that for any simple closed curve C directed positively,

$$\oint_C -y\,dx = \text{Area inside } C\,.$$

c) The curve $y^2 = x^2(1-x)$ shown is given parametrically by

$$x = 1 - t^2\,, \quad y = t - t^3\,.$$

Find the area inside the loop.



Problem 7 (15 points; 5, 10)

a) Write down in rectangular coordinates the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ whose vectors all have unit length and point radially outward from the origin. ($\mathbf{F} = 0$ at $(0,0)$.)

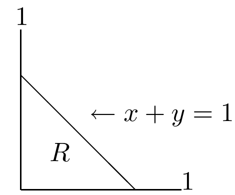
b) For this field, give the values of $\int_C \mathbf{F}\cdot d\mathbf{r}$ over the following curves (no calculation is required):

C_1 is the unit semi-circle in the upper half-plane, running from $(1,0)$ to $(-1,0)$

C_2 is the line segment from $(0,0)$ to $(1,1)$

Brief solutions.

Problem 1



Thus the volume is $\int_0^1 \int_0^{1-x} (1-x-y)\,dx\,dy$ OR $\int_0^1 \int_0^{1-4} (1-x-y)\,dx\,dy$.

Problem 2

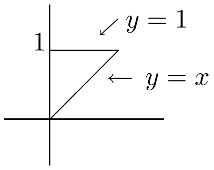
With the center on the x-axis

$$\text{Mass } = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \cdot r\,dr\,d\theta$$

OR with the center on the y-axis:

$$\text{Mass } = \int_0^{\pi} \int_0^{2a\sin\theta} r \cdot r\,dr\,d\theta$$

Problem 3 The region of integration for $\int_0^1 \int_x^1$ is

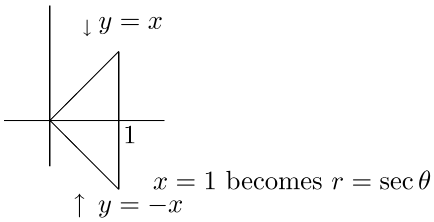


so the integral becomes $\int_0^1 \int_0^y \cos(y^2)\,dx\,dy$, which evaluates by

Inner: $\cos(y^2)\cdot x\big]_0^1 = \cos(y^2)\cdot y$

Outer: $\frac{1}{2}\sin(y^2)\big]_0^1 = \frac{1}{2}\sin 1$

Problem 4 The region of integration for $\int_0^1 \int_{-x}^{x} \dots\, dy\,dx$ is $(r^2 = x^2 + y^2)$

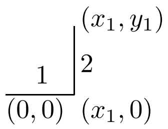


so in polar coordinates the integral becomes

$$\int_{-\pi/4}^{\pi/4} \int_0^{1\sec\theta)} r^3 \times r\,dr\,d\theta.$$

Problem 5

1. $\vec{F}$ conservative $\iff \frac{\partial(axy+y^2)}{\partial y} = \frac{\partial(x^2+bxy+1)}{\partial x} \iff ax + 2y = 2x + by \iff a = 2,\ b = 2$
2. **Method 1:**



$f(x_1, y_1) = \int_1 + \int_2 \qquad \int_1 = 0$ since along 1 $y = 0\ dy = 0$

$\int_2 = \int_0^y (x_1^2 + 2x_1 y + 1)\,dy = x_1^2 y_1 + x_1 y_1^2 + y_1$

since $x < x_1$, $dx = 0$ along path 2

OR

3. **Method 2:** $\frac{\partial f}{\partial x} = 2xy + y^2$

$$\begin{aligned} \therefore f &= x^2y + xy^2 + g(y) \\ \frac{\partial f}{\partial y} &= x^2 + 2xy + g'(y) \qquad = \quad x^2 + 2xy + 1 \\ &= g'(y) = 1, \quad g(y) = y \end{aligned}$$

and so

$$f = x^2y + xy^2 + y$$

4. Using fundamental theorems:

$$\int_{(x_1,0)}^{(x_0,0)} \vec{F}\cdot d\vec{r} = \int_{(x_1,0)}^{(x_0,0)} \nabla(x^2y + xy^2 + y)\cdot d\vec{r} = 0 - 0 = 0$$

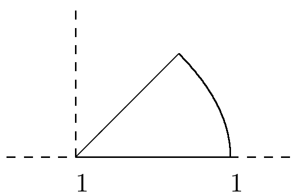
OR

Since $\vec{F}$ is path-independent, we can replace C by a path D on the x-axis:

$$\int_C \vec{F}\cdot d\vec{r} = \int_D \vec{F}\cdot d\vec{r} = \int_{x_1}^{x_2} 0\,(\text{since } y = 0 \text{ on } D)\cdot\, dx = 0.$$

Problem 6

1. $\oint_C -x^2y\,dx + xy^2\,dy = \iint_R y^2 - (-x^2)\,dA$ by Green's theorem



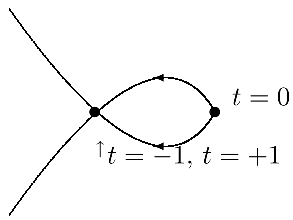
$$= \int_0^{\pi/4}\int_0^1 r^2\cdot r\,dr\,d\theta = \frac{\pi}{4}\cdot\frac{r^4}{4}\Big]_0^1 = \frac{\pi}{16}.$$

The original picture can be interpreted to mean that C was not closed, with the part in the x-axis missing. Since this actually contributes nothing to the line integral, the answer is the same!

2. By Green's theorem, $\oint_C -y\,dx = \iint_R -\frac{\partial(-y)}{\partial y}\,dA = \iint_R dA =$ area of R
3. Using (b), area inside loop

$$= \oint -ydx = \int_{-1}^{1} -(t-t^3)(-2t)\,dt = \int_{-1}^{1}(2t^2 - 2t^4)\,dt$$

$$= \frac{2t^3}{3} - \frac{2t^5}{5}\Big]_{-1}^{1} = \frac{2}{3} - \frac{2}{5} - (\frac{-2}{3} + \frac{2}{5}) = 8/15$$



Problem 7

a) $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ has magnitude $= 1$ (since $|x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2+y^2}$) and has radially outward direction.

b) For C_1, the unit semi-circle in the upper half-plane, running from $(1, 0)$ to $(-1, 0)$, $\mathbf{F}$ is perpendicular to the tangent of the curve, at every point, so the integral is zero.

For C_2, $\mathbf{F}$ and C have the same direction and both are constant (on C_2), so the work is simply $|\mathbf{F}| \times$ (distance) $= 1 \cdot \sqrt{2} = \sqrt{2}$.