

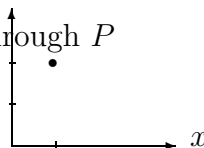
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EXAM 2 FOR 18.02, SPRING 1999

For full credit try all problems. Write your name on each page and try if possible to do all your work on these pages. If it is necessary to add some more pages, write your name on each.

Problem 1 Consider the function $w = f(x, y) = y^2/x^3$ near the point $P = (1, 2)$.

- a) Sketch the level curve of w through P



- b) Compute the gradient $\vec{\nabla}w$ at P .
 c) Give an approximate formula for Δw , the change in w , arising from small changes Δx and Δy in x and y starting from P .
 d) To changes in which of the variables x or y is w more sensitive at P ?
 e) Find the directional derivative $\frac{dw}{ds}|_P$ in the direction of $\langle -1, 1 \rangle$.

Solution:

- a) Accept any reasonable attempt here, it only needs to have the right derivative (or arguably so) at P .
 b) $\vec{\nabla}w = -3\frac{y^2}{x^4}\mathbf{i} + 2\frac{y}{x^3}\mathbf{j} = -12\mathbf{i} + 4\mathbf{j}$ at P . Accept vector in any identifiable form.
 c) $\Delta w \simeq \vec{\nabla}w|_P \cdot \Delta\vec{x} = -12\Delta x + 4\Delta y$.
 d) It is more sensitive to changes in x . Don't penalize earlier computational errors if the argument is correct.
 e) $\frac{dw}{ds}|_{P,\hat{e}} = \vec{\nabla}w \cdot \hat{e} = (12 + 4)/\sqrt{2} = 8\sqrt{2}$.

Problem 2 Consider the surface defined by $xyz = 2$.

- a) Find the tangent plane at $\langle 1, 1, 2 \rangle$ —write it in the form

$$ax + by + cz = d.$$

- b) Find the point on the surface at which the normal has the same direction as $\mathbf{i} + 2\mathbf{j} + \frac{1}{4}\mathbf{k}$.

Solution:

- a) The gradient is $\vec{\nabla}(xyz) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ which is $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ at $\langle 1, 1, 2 \rangle$. Thus the tangent plane is of the form $2(x - 1) + 2(y - 1) + (z - 2) = 0$, that is $2x + 2y + z = 6$.

- b) For the normal to have the same direction we must have $yz = \lambda$, $xz = 2\lambda$, $xy = \frac{1}{4}\lambda$ for some constant λ . Since $xyz = 2$ this implies $\frac{1}{2}\lambda^3 = 4$ so $\lambda = 2$ and $\langle x, y, z \rangle = \langle 1, \frac{1}{2}, 4 \rangle$. This is the only point on the surface with the stated property.

Problem 3 Find *all* critical points of the function

$$f(x, y) = 4x^2 + 2y^2 + 3xy^2 + 1.$$

Work out the type (max/min/saddle/other) of each of them.

Solution: $\partial f/\partial x = 8x + 3y^2$, $\partial f/\partial y = 4y + 6xy$, $\partial^2 f/\partial x^2 = 8$, $\partial^2 f/\partial x\partial y = 6y$, $\partial^2 f/\partial y^2 = 4 + 6x$. Critical points therefore satisfy $8x = -3y^2$, $2y(2 + 3x) = 0$. Thus either $y = 0$ or $3x = -2$. The first gives $x = 0, y = 0$. The second gives $16 = 9y^2$ so $y = \pm 4/3$. Thus there are three critical points, at $(0, 0)$ and $(-2/3, \pm 4/3)$. The discriminant, $\partial^2 f/\partial x^2 \cdot \partial^2 f/\partial y^2 - (\partial^2 f/\partial x\partial y)^2$ at these points is respectively $8 \cdot 4 > 0$, $8 \cdot (4 - 4) - (\pm 4/3)^2 < 0$ so $(0, 0)$ is a local minimum, and $(-2/3, \pm 4/3)$ are both saddle points.

Problem 4 Suppose that f and g are (nice) functions of one variable. Show that the function of two variables

$$z = f(x + y) + g(x - y)$$

satisfies $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ (the wave equation).

Solution: By the chain rule, $\partial z/\partial x = f'(x + y) + g'(x - y)$, $\partial^2 z/\partial x^2 = f''(x + y) + g''(x - y)$ and similarly $\partial z/\partial y = f'(x + y) - g'(x - y)$ so $\partial^2 z/\partial y^2 = f''(x + y) + g''(x - y) = \partial^2 z/\partial x^2$ as claimed.

Problem 5 Find the maximum and minimum values of the function $f(x, y) = 4x + 2y + 3$ on the ellipse $2x^2 + y^2 = 3$.

Solution: Using Lagrange's method, find the critical points of $f(x, y) - \lambda g(x, y)$ with $g(x, y) = 2x^2 + y^2 - 3$. Thus $4 + \lambda(4x) = 0 = 2 + \lambda(2y)$ so $x\lambda = -1 = \lambda y$. So $\lambda = 0$ or $x = y$. The former case cannot occur, so $x = y$ hence $3x^2 = 3$, $x = y = \pm 1$. At these two points $f = 9$ and $f = -3$ respectively, so the maximum is 9 and the minimum is -3. Maybe they can make substitution work, in which case it should be allowed.

Problem 6 Find the point of intersection of the two planes $x + y - z = 1$ and $x + 3y - 2z = 2$ which is closest to the origin.

Solution: Substitution is not bad - on the intersection $1 - y + z = 2 - 3y + 2z$ so $2y - 1 = z$ and $x = 1 - y + z = 1 - y + 2y - 1 = y$. Thus the square of the distance is $x^2 + y^2 + z^2 = 2y^2 + (2y - 1)^2$ with the minimum at $x = y = \frac{1}{3}, z = -\frac{1}{3}$.