EXAM 1 FOR 18.02, SPRING 1999
YOUR NAME:
RECITATION INSTRUCTOR
RECITATION TIME

For full credit try all problems. Write your name on each page and try if possible to do all your work on these pages. If it is necessary to add some more pages, write your name on each.

Marks:

1.
2.
3.
4.
5.

Problem 1 (20 points; 7, 6, 6) Let $\mathbf{A} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$, $\mathbf{B} = 2\hat{\imath} + 3\hat{\jmath} + c\hat{k}$.

1. For what values of c will $\mathbf{A}$ and $\mathbf{B}$ be perpendicular?
2. For what values of c will $\mathbf{A}$ and $\mathbf{B}$ be parallel?
3. If $c = 1$ find $\cos\theta$ where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

Solution

1. The dot product is $\mathbf{A}\cdot\mathbf{B} = 4 - 9 + c$ which vanishes exactly when $c = 5$. So $\mathbf{A}$ and $\mathbf{B}$ are perpendicular when $c = 5$ only.
2. The cross product is $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 2 & 3 & c \end{vmatrix} = (-3c - 3)i - (2c - 2)j + (6+6)k$ can never vanish, so $\mathbf{A}$ and $\mathbf{B}$ are not perpendicular for any value of c. (Nasty huh!)
3. If $c = 1$ then $\mathbf{B} = 2i + 3j + k$. Both $\mathbf{A}$ and $\mathbf{B}$ have length $\sqrt{14}$ and $\mathbf{A}\cdot\mathbf{B} = -4$ so $\cos\theta = -4/14 = -2/7$.

Problem 2 (20 points;7,7,6) Consider the three points $P = (1, 0, 0)$, $Q = (2, 2, 0)$ and $R = (2, 0, 1)$.

1. Find the area of the triangle with these points as vertices.
2. Give the equation of the plane through these three points in the form $ax + by + cz = d$.
3. Find parametric equations for the line which passes through the point P and is perpendicular to the plane through P, Q and R.

Solution

1. The area of the triangle is $\frac{1}{2}|\mathbf{PQ} \times \mathbf{PR}|$. Since $\mathbf{PQ} = i + 2j$ and $\mathbf{PR} = i + k$, $\mathbf{PQ} \times \mathbf{PR} = \mathbf{2i} - \mathbf{j} - \mathbf{2k}$. Thus the area of the triangle is $3/2$.
2. Since the normal to the plane is $\mathbf{PQ} \times \mathbf{PR}$ it is given by the equation $2x - y - 2z = 2$.
3. This line has direction $\mathbf{PQ} \times \mathbf{PR}$ so has parametric equations $x = 2t + 1$, $y = -t$ and $z = -2t$. (Other forms are possible.)

Problem 3 (25 points; 5, 10, 5, 5) Consider the system of equations

$$\begin{aligned} x + y - z &= 1 \\ 3x - 4z &= -1 \\ 2x + ay &= 9. \end{aligned}$$

1. Write these equations in matrix form $\mathbf{AX} = \mathbf{Y}$ where $\mathbf{A}$ is a matrix and $\mathbf{X}$ and $\mathbf{Y}$ are column vectors.
2. Set $a = 2$ and find $\mathbf{A}^{-1}$.
3. Use your computation of $\mathbf{A}^{-1}$ to solve the equations when $a = 2$.
4. For which value(s) of a does the homogeneous equation $\mathbf{AX} = 0$ *not* have a unique solution?

Solution

1. These equations are of the form $\mathbf{AX} = \mathbf{Y}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 0 & -4 \\ 2 & a & 0 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \mathbf{Y} = \begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}.$$

2. If $a = 2$ the determinant is

$$\begin{vmatrix} 1 & 1 & -1 \\ 3 & 0 & -4 \\ 2 & a & 0 \end{vmatrix} = 0 - 8 - 3a - 0 - 0 + 4a = a - 8 = -6.$$

Then successively the minor, cofactor, transpose cofactor and inverse matrices are

$$\begin{bmatrix} 8 & 8 & 6 \\ 2 & 2 & 0 \\ -4 & -1 & -3 \end{bmatrix}, \begin{bmatrix} 8 & -8 & 6 \\ -2 & 2 & 0 \\ -4 & 1 & -3 \end{bmatrix},$$
$$\begin{bmatrix} 8 & -2 & -4 \\ -8 & 2 & 1 \\ 6 & 0 & -3 \end{bmatrix}, \begin{bmatrix} -4/3 & 1/3 & 2/3 \\ 4/3 & -1/3 & -1/6 \\ -1 & 0 & 1/2 \end{bmatrix}.$$

3. The solution is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y} = \begin{bmatrix} 13/3 \\ 1/6 \\ 7/2 \end{bmatrix}$ so $x = 13/3$, $y = 1/6$ and $z = 7/2$.
4. The determinant vanishes only if $a = 8$ so for this value the homogeneous equation $\mathbf{AX} = 0$ does not have a unique solution.

Problem 4 (20 points; 5, 5, 5, 5) The coordinates of a moving point satisfy the parametric equations

$$x = 2t, y = 4 - t, z = t^2 + 1.$$

1. At what point does it meet the plane $x + y = 2$?
2. Calculate the velocity at this point.
3. What is the acceleration at his point?
4. Find the point where the speed is smallest.

Solution

1. $x + y = 4 + t$ on the curve, so $t = -2$ and hence $x = -4$, $y = 6$, $z = 5$.
2. The velocity curve is $2\hat{\imath} - \hat{\jmath} + 2t\hat{k} = 2\hat{\imath} - \hat{\jmath} - 4\hat{k}$ at the point of intersection with the plane.
3. The acceleration is constant and is $2\hat{k}$.
4. The square of the speed for any t is $4 + 1 + 4t^2$ which takes its minimum value 5 at $t = 0$. Thus the point at which the speed is smallest is $(0, 4, 1)$.

Problem 5 (15 points) Using vector methods show that the midpoints of the sides of any quadrilateral in the plane are the corners of a parallelogram.

Solution This is a proof from class. If the vertices in clockwise order are P, Q, R and T then four sides of the quadrilateral with vertices the midpoints of the sides of the given quadrilateral are, in order, $\frac{1}{2}(\mathbf{PQ} + \mathbf{QR})$, $\frac{1}{2}(\mathbf{QR} + \mathbf{RT})$, $\frac{1}{2}(\mathbf{RT} + \mathbf{TP})$ and $\frac{1}{2}(\mathbf{TP} + \mathbf{PQ})$. Since $\mathbf{PQ} + \mathbf{QR} = \mathbf{PT} + \mathbf{TR}$ the first and third sides sum to zero and similarly the second and last sum to zero, so this is a parallelgram.