Practice Exam 1 for 18.02, Spring 1999

Problem 1 (25 points; 10, 10, 5) Let $\mathbf{A} = \mathbf{i} + \mathbf{2j} + \mathbf{2k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

1. Find a vector perpendicular to both **A** and **B**.
2. Find the equation of the plane passing through the point$(1, 0, 8)$ and parallel to both **A** and **B**.
3. Determine where the plane meets the y-axis.

Solution

1. $\mathbf{N} = \mathbf{A} \times \mathbf{B} = j - k$ is perpendicular to both.
2. The plane is $\mathbf{N} \cdot \mathbf{x} = d$ so $y - z = -8$.
3. It meets the y-axis (which is $x = 0, z = 0$) at the point $(0, -8, 0)$.

Problem 2 (15 points) Use vector methods to show that the line joining the mid-points of two sides of a triangle is parallel to the third side.

Solution If the vertices of the triangle are P, Q and R then the sides are **PQ**, **QR** and **RP**. The vector from the midpoint of PQ to the midpoint of QR is $\frac{1}{2}(\mathbf{PQ} + \mathbf{QR})$. Summing the three sides gives zero so $\mathbf{PQ} + \mathbf{QR} + \mathbf{RP} = 0$ which implies that $\frac{1}{2}(\mathbf{PQ} + \mathbf{QR}) = -\frac{1}{2}\mathbf{RP}$, so the line between the midpoints of two sides is parallel to the base **RP**.

Problem 3 (25 points; 20, 5) Consider the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + cx_3 &= y_1 \\ -x_1 + x_3 &= y_2 \\ x_1 + x_2 + x_3 &= y_3 \end{aligned}$$

1. Take $c = 3$. Write the system in matrix form $\mathbf{Ax} = \mathbf{y}$. Calculate $\mathbf{A^{-1}}$ and use it to find equations expressing x_1, x_2, x_3 in terms of y_1, y_2 and y_3.
2. For what value(s) of c is it *not* possible to solve for the x's in terms of the y's?

Solution

1. The matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 & c \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. For $c = 3$ the cofactor matrix of this is $\begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 3 & -5 & 3 \end{bmatrix}$. The determinant (in general) is $3 - c + 3 - 2 = 4 - c = 1$ when $c = 3$. Thus the inverse matrix is

$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & -5 \\ -1 & 1 & 3 \end{bmatrix}$. It follows that $\mathbf{x} = \mathbf{A} \cdot \mathbf{y}$ so

$$x_1 = -y_1 + 3y_3,\ x_2 = 2y_1 - y_2 - 5y_3,\ x_3 = -y_1 + y_2 + 3y_3.$$

2. For $c = 4$ since the determinant of $\mathbf{A}$ vanishes and there is then either no solution or an infinite number of solutions depending on $\mathbf{y}$.

Problem 4 (25 points; 10, 5, 5, 5) The motion of a point $P = (x, y, z)$ in space is described by the parametric equations

$$x = 2 + t^2 \quad y = t + 1 \quad z = t^2 + 4t + 1\,.$$

1. Does the curve meet the plane $x + z = 0$?
2. Where does the curve meet the plane $y = 0$?
3. Compute the velocity vector for the curve.
4. Find the point at which the speed is smallest.

Solution

1. On the curve$x + z = 2t^2 + 4t + 3 = 2(t + 1)^2 + 1$ which never vanishes, so the curve does not meet the plane $x + z = 0$.
2. The velocity vector is $2t\hat{i} + \hat{j} + (2t + 4)\hat{k}$.
3. The square of the speed is $4t^2 + 1 + 4(t + 2)^2 = 8(t + 1)^2 + 9$ so the minimum occurs at $t = -1$, which means at the point $(3, 0, -2)$.

Problem 5 (10 points) Consider the vectors

$$\mathbf{A} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \mathbf{B} = \mathbf{i} + \mathbf{j} \text{ and } \mathbf{C} = \mathbf{i} - \mathbf{j} - \mathbf{2k}\,.$$

1. Show that each is perpendicular to the other two.
2. Find constants c_1, c_2, c_3 so that $\mathbf{i} = c_1\mathbf{A} + c_2\mathbf{B} + c_3\mathbf{C}$.

Solution

1. $\mathbf{A} \cdot \mathbf{B} = 1 - 1 = 0$, $\mathbf{B} \cdot \mathbf{C} = 1 - 1 = 0$, $\mathbf{A} \cdot \mathbf{C} = 1 + 1 - 2 = 0$ so each is perpendicular to the others.
2. The dot products are $\mathbf{i} \cdot \mathbf{A} = 1$, $\mathbf{i} \cdot \mathbf{B} = 1$, $\mathbf{i} \cdot \mathbf{C} = 1$. The squares of the lengths are $|\mathbf{A}|^2 = 3$, $|\mathbf{B}|^2 = 2$ and $|\mathbf{C}|^2 = 6$ so

$$\mathbf{i} = \frac{1}{3}\mathbf{A} + \frac{1}{2}\mathbf{B} + \frac{1}{6}\mathbf{C},\ c_1 = \frac{1}{3},\ c_2 = \frac{1}{2},\ c_3 = \frac{1}{6}.$$