A HALF-TWIST TYPE FORMULA FOR THE R-MATRIX OF A SYMMETRIZABLE KAC-MOODY ALGEBRA

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ABSTRACT. Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal R-matrix of $U_q(\mathfrak{g})$ of the form $R = (X^{-1} \otimes X^{-1})\Delta(X)$. The action of X on a representation V permutes weight spaces according to the longest element in the Weyl group, so is only defined when $\mathfrak{g}$ is of finite type. We give a similar formula which is valid for any symmetrizable Kac-Moody algebra. This is done by replacing the action of X on V with an endomorphism that preserves weight spaces, but which is bar-linear instead of linear.

1. Introduction

Let $\mathfrak{g}$ be a finite type complex simple Lie algebra, and let $U_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra. In [KR] and [LS], Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal R-matrix

$$R = (X^{-1} \otimes X^{-1})\Delta(X), \tag{1}$$

where X belongs to a completion of $U_q(\mathfrak{g})$. The element X is constructed using the braid group element T_{w_0} corresponding to the longest word of the Weyl group, so only makes sense when $\mathfrak{g}$ is of finite type.

The element X defines a vector space endomorphism X_V on each representation V, and in fact X is defined by this system $\{X_V\}$ of endomorphisms. With this point of view, Equation (1) is equivalent to the claim that, for any finite dimensional representations V and W and $u \in V \otimes W$,

$$R(u) = (X_V^{-1} \otimes X_W^{-1})X_{V\otimes W}(u). \tag{2}$$

In the present work we replace X_V with an endomorphism Θ_V which preserves weight spaces. We show that, for any symmetrizable Kac-Moody algebra $\mathfrak{g}$, and any integrable highest weight representations V and W of $U_q(\mathfrak{g})$, the action of the universal R-matrix on $u \in V \otimes W$ is given by

$$R(u) = (\Theta_V^{-1} \otimes \Theta_W^{-1})\Theta_{V\otimes W}(u). \tag{3}$$

There is a technical difficulty because Θ_V is not linear over the base field $\mathbb{Q}(q)$, but instead is compatible with the automorphism of $\mathbb{Q}(q)$ which inverts q. For this reason Θ_V depends on a choice of a "bar involution" on V. To make Equation (3) precise we define a bar involution on $V \otimes W$ in terms of chosen involutions of V and W, and then show that the composition $(\Theta_V^{-1} \otimes \Theta_W^{-1})\Theta_{V\otimes W}$ does not depend on any choices.

The system of endomorphisms Θ was previously studied in [T], where it was used to construct the universal R-matrix when $\mathfrak{g}$ is of finite type. Essentially we have extended this previous work to include all symmetrizable Kac-Moody algebras. However, the action of Θ on a tensor product is defined differently here than in [T], so the constructions of R are a-priori not identical, and we have not in fact proven that the construction in [T] gives the universal R-matrix in all cases.

This note is organized as follows. In Section 2 we establish notation and review some background material. In Section 3 we construct the system of endomorphisms Θ. In Section 4 prove our main

Theorem (Theorem 4.1), which simply says that our construction gives the universal R-matrix in all cases. In Section 5 we discuss two questions which motivated this work.

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2. Background

2.1. **Conventions.** We first fix some notation. For the most part we follow conventions from [CP].

- $\mathfrak{g}$ is a complex simple Lie algebra with Cartan algebra $\mathfrak{h}$ and Cartan matrix $A = (a_{ij})_{i,j \in I}$.
- $\langle \cdot, \cdot \rangle$ denotes the paring between $\mathfrak{h}$ and $\mathfrak{h}^\star$ and $(\cdot, \cdot)$ denotes the usual symmetric bilinear form on either $\mathfrak{h}$ or $\mathfrak{h}^\star$. Fix the usual bases α_i for $\mathfrak{h}^\star$ and H_i for $\mathfrak{h}$, and recall that $\langle H_i, \alpha_j \rangle = a_{ij}$.
- $d_i = (\alpha_i, \alpha_i)/2$, so that $(H_i, H_j) = d_j^{-1} a_{ij}$.
- ρ is the weight satisfying $(\alpha_i, \rho) = d_i$ for all i.
- $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra associated to $\mathfrak{g}$, generated over $\mathbb{Q}(q)$ by E_i and F_i for all $i \in I$, and K_w for w in the co-weight lattice of $\mathfrak{g}$. As usual, let $K_i = K_{H_i}$. We use conventions as in [CP]. For convenience, we recall the exact formula for the coproduct:

$$\begin{cases} \Delta E_i &= E_i \otimes K_i + 1 \otimes E_i \\ \Delta F_i &= F_i \otimes 1 + K_i^{-1} \otimes F_i \\ \Delta K_i &= K_i \otimes K_i \end{cases} \tag{4}$$

- We in fact need to adjoint a fixed k^{th} root of q to $\mathbb{Q}(q)$, where k is twice the size of the weight lattice mod the root lattice. We denote this by $q^{1/k}$.
- V_λ is the irreducible representation of $U_q(\mathfrak{g})$ with highest weight λ.
- v_λ is a highest weight vector of V_λ.
- A vector v in a representation V is called *singular* if $E_i(v) = 0$ for all $i \in I$.
- $V(\mu)$ denotes the μ weight space of V.
- Throughout, a representation of $U_q(\mathfrak{g})$ means a type 1 integrable highest weight representation.

2.2. **The R-matrix.** We briefly recall the definition of a universal R-matrix, and the related notion of a braiding.

Definition 2.1. *A braided monoidal category is a monoidal category $\mathcal{C}$, along with a natural system of isomorphisms $\sigma^{br}_{V,W} : V \otimes W \to W \otimes V$ for each pair $V, W \in \mathcal{C}$, such that, for any $U, V, W \in \mathcal{C}$, the following two equalities hold:*

$$\sigma^{br}_{U,W} \otimes Id \circ Id \otimes \sigma^{br}_{V,W} = \sigma^{br}_{U \otimes V, W} \tag{5}$$

$$Id \otimes \sigma^{br}_{U,W} \circ \sigma^{br}_{U,V} \otimes Id = \sigma^{br}_{U, V \otimes W}. \tag{6}$$

The system $\sigma^{br} := \{\sigma^{br}_{V,W}\}$ is called a braiding on $\mathcal{C}$.

Let $U_q(\widetilde{\mathfrak{g}) \otimes U_q}(\mathfrak{g})$ be the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in the weak topology defined by all matrix elements of $V_\lambda \otimes V_\mu$, for all ordered pairs of deminant integral weights (λ, μ).

Definition 2.2. *A universal R-matrix is an element R of $U_q(\widetilde{\mathfrak{g}) \otimes U_q}(\mathfrak{g})$ such that $\sigma^{br}_{V,W} := Flip \circ R$ is a braiding on the category of $U_q(\mathfrak{g})$ representations. Equivalently, an element R is a universal R-matrix if it satisfies the following three conditions*

(i) *For all $u \in U_q(\mathfrak{g})$, $R\Delta(u) = \Delta^{op}(u) R$.*
(ii) *$(\Delta \otimes 1)R = R_{13}R_{23}$, where R_{ij} mean R placed in the i and j^{th} tensor factors.*
(iii) *$(1 \otimes \Delta)R = R_{13}R_{12}$.*

The following theorem is central to the theory of quantized universal enveloping algebra. See [CP] for a discussion when $\mathfrak{g}$ is of finite type, and [L] for the general case.

Proposition 2.3. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Then $U_q(\mathfrak{g})$ has a unique universal R-matrix of the form*

$$R = A\Big(1 \otimes 1 + \sum_{\substack{\text{positive integral} \\ \text{weights } \beta \text{ (with} \\ \text{multiplicity)}}} X_\beta \otimes Y_\beta\Big), \tag{7}$$

where X_β has weight β, Y_β has weight $-\beta$, and for all $v \in V$ and $w \in W$, $A(v \otimes w) = q^{(\mathrm{wt}(v), \mathrm{wt}(w))}$.

2.3. **Constructing isomorphisms using systems of endomorphisms.** In this section we review a method for constructing natural systems of isomorphisms $\sigma_{V,W} : V \otimes W \to W \otimes V$ for representations V and W of $U_q(\mathfrak{g})$. This idea was used by Henriques and Kamnitzer in [HK], and was further developed in [KT2]. The data needed is:

(i) An algebra automorphism C_ξ of $U_q(\mathfrak{g})$ which is also a coalgebra anti-automorphism.

(ii) A natural system of invertible (vector space) endomorphisms ξ_V of each representation V of $U_q(\mathfrak{g})$ such that the following diagram commutes for all V:

$$\begin{array}{ccc} V & \xrightarrow{\quad \xi_V \quad} & V \\ \circlearrowleft & & \circlearrowleft \\ U_q(\mathfrak{g}) & \xrightarrow{\quad C_\xi \quad} & U_q(\mathfrak{g}). \end{array} \tag{8}$$

It follows immediately from the definition of coalgebra anti-automorphism that

$$\sigma^\xi := \mathrm{Flip} \circ (\xi_V^{-1} \otimes \xi_W^{-1}) \circ \xi_{V \otimes W} \tag{9}$$

is an isomorphism of $U_q(\mathfrak{g})$ representations from $V \otimes W$ to $W \otimes V$.

In the current work we require a little more freedom: we will sometimes use automorphisms C_ξ of $U_q(\mathfrak{g})$ which are not linear over $\mathbb{C}(q)$, but instead are bar-linear (i.e. invert q). This causes some technical difficulties, which we deal with in Section 3.

Comment 2.4. To describe the data $(\mathcal{C}_\xi, \xi)$, it is sufficient to describe C_ξ, and the action of ξ_{V_λ} on any one vector v in each irreducible representation V_λ. This is usually more convenient then describing ξ_{V_λ} explicitly. Of course, the choice of C_ξ imposes a restriction on the possibilities for $\xi_{V_\lambda}(v)$, so when we give a description of ξ in this way we are always claiming that the action on our chosen vector in each V_λ is compatible with C_ξ.

2.4. **A useful lemma.** Let (V_λ, v_λ) and (V_μ, v_μ) be irreducible representations with chosen highest weight vectors. Every vector $u \in V_\lambda \otimes V_\mu$ can be written as

$$u = v_\lambda \otimes c_0 + b_{k-1} \otimes c_1 + \ldots + b_1 \otimes c_{k-1} + b_0 \otimes v_\mu, \tag{10}$$

where, for $0 \le j \le k-1$, b_j is a weight vector of V_λ of weight strictly less then λ, and c_j a weight vector of V_μ of weight strictly less then μ. Furthermore, the vectors $b_0 \in V_\lambda$ and $c_0 \in V_\mu$ are uniquely determined by u. Thus we can define projections from $V_\lambda \otimes V_\mu$ to V_λ and V_μ as follows:

Definition 2.5. *The projections $p^1_{\lambda,\mu} : V_\lambda \otimes V_\mu \to V_\lambda$ and $p^2_{\lambda,\mu} : V_\lambda \otimes V_\mu \to V_\mu$ are given by, for all $u \in V_\lambda \otimes V_\mu$,*

$$p^1_{\lambda,\mu}(u) := b_0 \tag{11}$$

$$p^2_{\lambda,\mu}(u) := c_0. \tag{12}$$

Lemma 2.6. *Let $S_{\lambda,\mu}$ be the space of singular vectors in $V_\lambda \otimes V_\mu$. The restrictions of the maps $p^1_{\lambda,\mu}$ and $p^2_{\lambda,\mu}$ from Definition 2.5 to $S_{\lambda,\mu}$ are injective.*

Proof. We prove the Lemma only for $p^2_{\lambda,\mu}$, since the proof for $p^1_{\lambda,\mu}$ is completely analogous. Let $c_1, \cdots c_m$ be a weight basis for V_μ. Let u be a singular vector of $V_\lambda \otimes V_\mu$ of weight ν. Then u can be written uniquely as

$$u = \sum_{j=1}^{m} v_j \otimes c_j, \tag{13}$$

where each v_j is a weight vector in V_λ. Let γ be a maximal weight such that there is some j with $\mathrm{wt}(v_j) = \gamma$ and $v_j \neq 0$. It suffices to show that $\gamma = \lambda$, so assume for a contradiction that it does not. Then v_j is not a highest weight vector, so $E_i(v_j) \neq 0$ for some i. But then

$$E_i(u) = \sum_{\mathrm{wt}(v_{j_s})=\gamma} E_i(v_{j_s}) \otimes c_{j_s} + \begin{array}{l}\text{terms whose first factors have}\\ \text{weight strictly less then } \gamma + \alpha_i.\end{array} \tag{14}$$

Since the c_j are linearly independent and $E_i(v_j) \neq 0$ for some j with $\mathrm{wt}(v_j) = \gamma$, this implies that $E_i(u) \neq 0$, contradicting the fact that v is a singular vector. □

3. Constructing the system of endomorphisms Θ

Constructing and studying $\Theta = \{\Theta_V\}$ is the technical heart of this work. As we mentioned in the introduction, Θ_V is bar linear instead of linear, which makes it more difficult to choose a normalization. To get around this, we introduce the notion of a bar involution bar_V on V, and actualy define Θ on the category of representations with a chosen bar involution. We then define a tensor product on this new category, and show that $(\Theta^{-1}_{V,\mathrm{bar}_V} \otimes \Theta^{-1}_{W,\mathrm{bar}_W}) \circ \Theta_{(V,\mathrm{bar}_V)\otimes(W,\mathrm{bar}_W)}$ does not depend on the choices of bar_V and bar_W. The real work is in defining this tensor product, which essentially amounts to defining a bar involution on $V \otimes W$ in terms of bar involutions bar_V and bar_W.

3.1. Bar involution.

The following $\mathbb{Q}$ algebra involution of $U_q(\mathfrak{g})$ has been studied in several places, for example [K, Section 1.3], and is usually called bar involution. We use the notation C_{bar} because we will also work with bar involutions bar_V on representations V, which are compatible with C_{bar} in the sense of Equation (8).

Definition 3.1. *$C_{bar} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the $\mathbb{Q}$-algebra involution defined by*

$$\begin{cases} C_{bar} q = q^{-1} \\ C_{bar} K_i = K_i^{-1} \\ C_{bar} E_i = E_i \\ C_{bar} F_i = F_i. \end{cases}$$

It is perhaps useful to imagine that q is specialized to a complex number on the unit circle (although not a root of unity), so that C_{bar} is conjugate linear.

Definition 3.2. *Let V be a representation of $U_q(\mathfrak{g})$. A bar involution on V is a $\mathbb{Q}$-linear involution bar_V such that*

(i) *bar_V is compatible with C_{bar} in the sense that the following diagram commutes:*

$$
\begin{array}{ccc}
V & \xrightarrow{\quad bar_V \quad} & V \\
\circlearrowright & & \circlearrowright \\
U_q(\mathfrak{g}) & \xrightarrow{\quad \mathcal{C}_{bar} \quad} & U_q(\mathfrak{g}).
\end{array}
\tag{15}
$$

(ii) *Let $V^{inv} = \{v \in V$ such that $bar_V(v) = v\}$. Then $V = \mathbb{Q}(q) \otimes_{\mathbb{Q}} V^{inv}$.*

Comment 3.3. It is straightforward to check that C^2_{bar} is the identity. Along with condition (ii), this implies that bar^2_V is the identity, so the term "involution" is justified.

Comment 3.4. When it does not cause confusion we will denote $\mathrm{bar}_V(v)$ by $\bar{v}$.

Proposition 3.5. *Fix λ and a highest weight vector $v_\lambda \in V_\lambda$. There is a unique bar involution $bar_{(V_\lambda, v_\lambda)}$ on V_λ such that $bar_{(V_\lambda, v_\lambda)}(v_\lambda) = v_\lambda$.*

Proof. Recall that V_λ has a basis consisting of various $F_{i_k} \cdots F_{i_1} v_\lambda$. All of these vectors must be fixed by any bar involution preserving v_λ, so there is at most one possibility. On the other hand, it is clear that the unique $\mathbb{Q}$-linear map sending $f(q) F_{i_k} \cdots F_{i_1} v_\lambda$ to $f(q^{-1}) F_{i_k} \cdots F_{i_1} v_\lambda$ for each of these basis vectors is a bar involution. □

Corollary 3.6. *Every representation V has a (non-unique) bar involution bar_V.*

Proof. Choose a decomposition of V into irreducible components, and a highest weight vector in each irreducible component, then use Proposition 3.5. □

Definition 3.7. *Fix (V, bar_V) and (W, bar_W), where bar_V and bar_W are involutions of V and W compatible with C_{bar}. Let $(bar_V \otimes bar_W)$ be the vector space involution on $V \otimes W$ defined by $f(q) v \otimes w \to f(q^{-1}) \bar{v} \otimes \bar{w}$ for all $f(q) \in \mathbb{Q}(q)$ and $v \in V, w \in W$.*

Comment 3.8. It is straightforward to check that the action of $(\mathrm{bar}_V \otimes \mathrm{bar}_W)$ on a vector in $V \otimes W$ does not depend on its expression as a sum of elements of the form $f(q) v \otimes w$. The resulting map is a $\mathbb{Q}$-linear involution.

Definition 3.9. *Fix $u \in V_\lambda \otimes V_\mu$ a weight vector of weight ν. Define v^β for each weight β as the unique element of $V_\lambda(\nu - \beta) \otimes V_\mu(\beta)$ such that*

$$
u = \sum_{\textit{weights } \beta} v^\beta. \tag{16}
$$

Lemma 3.10. *Fix $(V_\lambda, bar_{V_\lambda})$ and (V_μ, bar_{V_μ}). Let v_ν be a singular weight vector in $V_\lambda \otimes V_\mu$, and write*

$$
v_\nu = \sum_{j=1}^{N} b_j \otimes c_j, \tag{17}
$$

where each b_j is a weight vector of V_λ, and each c_j is a weight vector of V_μ. Then

$$
bar(v_\nu) := \sum_{j=0}^{N} q^{(\mu,\mu) - (\mathrm{wt}(c_j), \mathrm{wt}(c_j)) + 2(\mu - \mathrm{wt}(c_j), \rho)} \bar{b}_j \otimes \bar{c}_j \tag{18}
$$

is also singular.

Proof. Fix $i \in I$. The vector v_ν is singular, so $E_i v_\nu = 0$ and hence $(E_i v_\nu)^\beta = 0$ for all β. Then:

$$0 = (E_i v_\nu)^\beta = \sum_{\mathrm{wt}(c_j)=\beta} q^{(\beta,\alpha_i)} E_i b_j \otimes c_j + \sum_{\mathrm{wt}(c_j)=\beta-\alpha_i} b_j \otimes E_i c_j. \tag{19}$$

Using Equation (18):

$$\begin{aligned}
(E_i \mathrm{bar}(v_\nu))^\beta = & \sum_{\mathrm{wt}(c_j)=\beta} q^{(\mu,\mu)-(\beta,\beta)+2(\mu-\beta,\rho)} q^{(\beta,\alpha_i)} E_i \bar{b}_j \otimes \bar{c}_j \\
& + \sum_{\mathrm{wt}(c_j)=\beta-\alpha_i} q^{(\mu,\mu)-(\beta-\alpha_i,\beta-\alpha_i)+2(\mu-\beta+\alpha_i,\rho)} \bar{b}_j \otimes E_i \bar{c}_j && (20) \\
& = q^{(\mu,\mu)-(\beta-\alpha_i,\beta-\alpha_i)+2(\mu-\beta+\alpha_i,\rho)} \times && (21) \\
& \quad \times \left(\sum_{\mathrm{wt}(c_j)=\beta} q^{-(\beta,\alpha_i)} E_i \bar{b}_j \otimes \bar{c}_j + \sum_{\mathrm{wt}(c_j)=\beta-\alpha_i} \bar{b}_j \otimes E_i \bar{c}_j \right) \\
& = q^{(\mu,\mu)-(\beta-\alpha_i,\beta-\alpha_i)+2(\mu-\beta+\alpha_i,\rho)} (\mathrm{bar}_{V_\lambda} \otimes \mathrm{bar}_{V_\mu})(E_i v_\nu)^\beta, && (22)
\end{aligned}$$

where $(\mathrm{bar}_{V_\lambda} \otimes \mathrm{bar}_{V_\mu})$ is the involution from Definition 3.7. But $E_i(v_\nu)^\beta = 0$, so we see that $E_i(v_\nu)^\beta = 0$. Since this holds for all i and all β, $\mathrm{bar}(v_\nu)$ is singular. □

Definition 3.11. *Let* $bar_{(V_\lambda,v_\lambda)\otimes(V_\mu,v_\mu)}$ *be the unique involution on* $V_\lambda \otimes V_\mu$ *which agrees with the involution bar from Lemma 3.10 on singular vectors, and is compatible with* C_{bar}.

Lemma 3.12. $bar_{(V_\lambda,v_\lambda)\otimes(V_\mu,v_\mu)}$ *is a bar involution.*

Proof. Definition 3.2 part (i) follows immediately from the definition of $\mathrm{bar}_{(V_\lambda,v_\lambda)\otimes(V_\mu,v_\mu)}$. To establish Definition 3.2 part (ii), it suffices to show that there is a basis for the space $S_{\lambda,\mu}$ of singular vetors of $V_\lambda \otimes V_\mu$ which is fixed by $\mathrm{bar}_{(V_\lambda,v_\lambda)\otimes(V_\mu,v_\mu)}$. Since $V_\lambda = \mathbb{Q}(q) \otimes_{\mathbb{Q}} V_\lambda^{inv}$, there is a basis for $S_{\lambda,\mu}$ consisting of elements of $V_\lambda^{inv} \otimes_{\mathbb{Q}} V_\mu$. Using Lemma 2.6, we see that there is a basis for $S_{\lambda,\mu}$ consisting of vectors of the form

$$v_\lambda \otimes c_0 + \cdots + b_0 \otimes v_\mu, \tag{23}$$

where $\bar{b}_0 = b_0$ and the missing terms are all of the form $b \otimes c$ with $\mathrm{wt}(c) < \mu$. By Definition 3.11 and Lemma 2.6, this vector is invariant under $\mathrm{bar}_{(V_\lambda,v_\lambda)\otimes(V_\mu,v_\mu)}$. □

In light of Definition 3.2 part (ii), we can extend Definition 3.11 by naturality to construct a bar-involution on $(V, \mathrm{bar}_V) \otimes (W, \mathrm{bar}_W)$ in terms of any bar-involutions bar_V and bar_W.

3.2. **The system of endomorphisms Θ.** Consider the $\mathbb{Q}$-algebra automorphism C_Θ of $U_q(\mathfrak{g})$:

$$\begin{cases} C_\Theta(E_i) = E_i K_i^{-1} \\ C_\Theta(F_i) = K_i F_i \\ C_\Theta(K_i) = K_i^{-1} \\ C_\Theta(q) = q^{-1}. \end{cases} \tag{24}$$

Notice that C_Θ is not linear over $\mathbb{Q}(q)$, but instead inverts q. One can easily check that C_Θ is a $\mathbb{Q}$ algebra involution, and that it is also a coalgebra anti-involution.

Definition 3.13. *Fix a representation* V *with a bar involution* bar_V. *Then* Θ_{V,bar_V} *is the* $\mathbb{Q}$ *linear endomorphism of* V *defined by*

$$\Theta_{V,bar_V}(v) = q^{-(\mathrm{wt}(v),\mathrm{wt}(v))/2+(\mathrm{wt}(v),\rho)} bar_V(v). \tag{25}$$

Comment 3.14. Using Definitions 3.1, one can see that, for any irreducible $V_\lambda \subset V$, $\Theta_{V,\mathrm{bar}_V}$ restricts to an endomorphism of V_λ.

Comment 3.15. There are sometimes weights λ for which $-(\lambda,\lambda)/2 + (\lambda,\rho)$ is not an integer. However, it is always a multiple of $1/k$ where k is twice the size of the weight lattice mod the root lattice. It is for this reason that we adjoin $q^{1/k}$ to the base field.

Lemma 3.16. *the following diagram commutes*

$$\begin{array}{ccc} V & \xrightarrow{\Theta_V} & V \\ \circlearrowright & & \circlearrowright \\ U_q(\mathfrak{g}) & \xrightarrow{C_\Theta} & U_q(\mathfrak{g}). \end{array} \tag{26}$$

Proof. It is sufficiant to check that $C_\Theta(X)\Theta_V(v) = \Theta_V(Xv)$, where $X = E_i$ or F_i. We do the case of F_i and leave E_i as an excersize. Fix $v \in V$.

$$\Theta_V(F_i v) = q^{-(\mathrm{wt}(F_i v),\mathrm{wt}(F_i v))/2+(\mathrm{wt}(F_i v),\rho)}\mathrm{bar}_V(F_i v) \tag{27}$$

$$= q^{-(\mathrm{wt}(v)-\alpha_i,\mathrm{wt}(v)-\alpha_i)/2+(\mathrm{wt}(v)-\alpha_i,\rho)}F_i\mathrm{bar}_V(v) \tag{28}$$

$$= q^{(\alpha_i,\mathrm{wt}(v)-\alpha_i)}q^{-(\mathrm{wt}(v),\mathrm{wt}(v))/2+(\mathrm{wt}(v),\rho)}F_i\mathrm{bar}_V(v) \tag{29}$$

$$= K_iF_iq^{-(\mathrm{wt}(v),\mathrm{wt}(v))/2+(\mathrm{wt}(v),\rho)}\mathrm{bar}_V(v) \tag{30}$$

$$= C_\Theta(F_i)\Theta_V(v). \tag{31}$$

where for Equation (29) we have used the fact that $(\alpha_i,\alpha_i)/2 = (\alpha_i,\rho) = d_i$. $\square$

Definition 3.17. *Fix two representations with bar involutions* (V, bar_V) *and* (W, bar_W)*. We set* $\Theta_{(V,bar_V)\otimes(W,bar_W)}$ *to be the* $\mathbb{Q}$ *linear endomorphism of* $V \otimes W$ *defined by, for all* $u \in V \otimes W$,

$$\Theta_{(V,bar_V)\otimes(W,bar_W)}(u) = q^{-(\mathrm{wt}(u),\mathrm{wt}(u))/2+(\mathrm{wt}(u),\rho)}\,bar_{(V\,bar_V)\otimes(W,bar_W)}. \tag{32}$$

Comment 3.18. By Lemma 3.12, $\mathrm{bar}_{(V\,\mathrm{bar}_V)\otimes(W,\mathrm{bar}_W)}$ is a bar involution on $V\otimes W$, so by Lemma 3.16, $\Theta_{(V,\mathrm{bar}_V)\otimes(W,\mathrm{bar}_W)}$ is compatible with C_Θ.

4. Main Theorem

Theorem 4.1. $(\Theta^{-1}_{V,bar_V} \otimes \Theta^{-1}_{W,bar_W})\Theta_{(V\otimes bar_V)\otimes(W,bar_W)}$ *acts on* $V\otimes W$ *as the standard R-matrix. This holds independent of the choice of bar involutions* bar_V *and* bar_W.

Proof. We will actually prove the equivalent statement that

$$\sigma^\Theta := \mathrm{Flip}\circ(\Theta^{-1}_{V,\mathrm{bar}_V} \otimes \Theta^{-1}_{W,\mathrm{bar}_W})\Theta_{(V\otimes\mathrm{bar}_V)\otimes(W,\mathrm{bar}_W)} \tag{33}$$

acts on $V\otimes W$ as the standard braiding $\mathrm{Flip}\circ R$. By Lemma 3.16 and the fact that C_Θ is a $\mathbb{Q}$ coalgebra anti-automorphism, the following diagram commutes:

$$\begin{array}{ccccc} V\otimes W & \xrightarrow{\Theta_{(V\otimes\mathrm{bar}_V)\otimes(W,\mathrm{bar}_W)}} & V\otimes W & \xrightarrow{\mathrm{Flip}\circ(\Theta^{-1}_{V\otimes\mathrm{bar}_V}\otimes\Theta^{-1}_{W,\mathrm{bar}_W})} & W\otimes V \\ \circlearrowright & & \circlearrowright & & \circlearrowright \\ U_q(\mathfrak{g}) & \xrightarrow{C_\Theta} & U_q(\mathfrak{g}) & \xrightarrow{C_\Theta^{-1}} & U_q(\mathfrak{g}). \end{array} \tag{34}$$

In particular, $\sigma^\Theta : V\otimes W \to W\otimes V$ is an isomorphism. Thus it suffices to show that $\sigma^\Theta(v_\nu) = \mathrm{Flip}\circ R(v_\nu)$ for every singular weight vector $v_\nu \in V\otimes W$. By naturality it is enough to consider the case when V and W are irreducible, so let v_ν be a singular vector in $V_\lambda\otimes V_\mu$. Write

$$v_\nu = b_\lambda\otimes c_0 + b_{k-1}\otimes c_1 + \ldots + b_1\otimes c_{k-1} + b_0\otimes b_\mu, \tag{35}$$

where for $0 \le j \le k-1$, b_j is a weight vector of V_μ of weight strictly less then μ. By Definitions 3.11 and 3.13,

$$\sigma^\Theta(v_\nu) = \text{Flip} \circ (\Theta^{-1}_{V_\lambda, \text{bar}_{V_\lambda}} \otimes \Theta^{-1}_{V_\mu, \text{bar}_{V_\mu}})\Theta_{(V_\lambda, \text{bar}_{V_\lambda}) \otimes (V_\mu, \text{bar}_{V_\mu})}(\cdots + b_0 \otimes b_\mu) \tag{36}$$

$$= \text{Flip} \circ (\Theta^{-1}_{V_\lambda, \text{bar}_{V_\lambda}} \otimes \Theta^{-1}_{V_\mu, \text{bar}_{V_\mu}})(q^{-(\mu+\text{wt}(b_0),\mu+\text{wt}(b_0))/2+(\mu+\text{wt}(b_0),\rho)}(\ldots + \bar{b}_0 \otimes \bar{b}_\mu)) \tag{37}$$

$$= q^{-(\text{wt}(b_0),\text{wt}(b_0))/2-(\mu,\mu)/2+(\mu+\text{wt}(b_0),\mu+\text{wt}(b_0))/2} b_\mu \otimes b_0 + \ldots \tag{38}$$

$$= q^{(\text{wt}(b_0),\mu)} b_\mu \otimes b_0 + \ldots, \tag{39}$$

where $\ldots$ always represents terms where the factor coming from V_μ has weight strictly less then μ. It follows immediately from Proposition 2.3 that

$$\text{Flip} \circ R(v_\nu) = q^{(\text{wt}(b_0),\mu)} b_\mu \otimes b_0 + \ldots, \tag{40}$$

where again $\ldots$ represents terms of the form $c \otimes b$ where $\text{wt}(c) < \mu$. Both $\sigma^\Theta(v_\nu)$ and $\text{Flip} \circ R(v_\nu)$ are singular vectors in $V_\mu \otimes V_\lambda$, so by Lemma 2.6 they are equal. □

Comment 4.2. The above proof works independent of the choice of bar_V and bar_W. One can also see directly that σ^Θ does not depend on these choices. Restrict to the irreducible case, and notice that by Lemma 3.5, σ^Θ depends only the a choice of highest weight vectors v_λ and v_μ. It is straightforward to check that rescaling these vectors has no effect on σ^Θ.

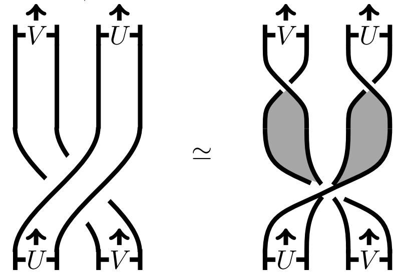
Comment 4.3. One can check that Θ_V is an involution of $\mathbb{Q}$ vector spaces, so the inverses in the statement of Theorem 4 are in some sense unnecessary. We include them because Θ_V should really be thought of as an isomorphism between V and the module which is V as a $\mathbb{Q}$ vector space, but with the action of $U_q(\mathfrak{g})$ twisted by C_Θ. We have not specified the action of Θ on this new module. The way the formula is written, Θ is always acting on V, W or $V \otimes W$ with the usual action, where it has been defined.

5. Future directions

We have two main motivations for developing our formula for the R-matrix.

Motivation 1. In work with Joel Kamnitzer [KT2], we showed that Drinfeld's unitarized R-matrix $\bar{R}$ (see [D]) respects crystal basis (up to some signs). Composing with Flip, we see that $\bar{R}$ descends to a crystal map from $B \otimes C$ to $C \otimes B$, which is fact agrees with the crystal commutor defined in [HK]. We make extensive use of Equation (1), so our methods are only valid in the finite type case. However Drinfeld's unitarized R-matrix is defined in the symmetrizable Kac-Moody case, as is the crystal commutor (see [KT1] and [S]). We hope that the formula given in Theorem 4.1 will help us to extend some of the results in [KT2] to the symmetrizable Kac-Moody case.

Motivation 2. Recall that the action of the braiding $\text{Flip} \circ R$ on $V \otimes W$ can be drawn diagrammatically as passing a string labeled V over a string labeled W. If we use flat ribbons in place of strings, as it is often convenient to do, one can consider the following isotopy:



Roughly, if one interprets twisting a ribbon by 180 degrees as X, and twisting two ribbon together as at the bottom on the right side as $\text{Flip} \circ \Delta(X)$, the two sides of this isotopy correspond to the two sides of Equation (1), written as

$$\text{Flip} \circ R = \text{Flip} \circ (X^{-1} \otimes X^{-1})\Delta(X) = (X^{-1} \otimes X^{-1}) \circ \text{Flip} \circ \Delta(X). \tag{41}$$

In work with Noah Snyder [ST], we make this precise. One should be able to use our new formula to give a precise interpretation of "twisting a ribbon by 180 degrees" in the symmetrizable Kac-Moody case. It is for this reason that we use the term "half twist type formula" in our title.

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