A HALF-TWIST TYPE FORMULA FOR THE R-MATRIX OF A
SYMMETRIZABLE KAC-MOODY ALGEBRA

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ABSTRACT. Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal R-matrix of $U_q(g)$ of the form $R = (X^{-1} \otimes X^{-1}) \Delta(X)$. The action of $X$ on a representation $V$ permutes weight spaces according to the longest element in the Weyl group, so is only defined when $g$ is of finite type. We give a similar formula which is valid for any symmetrizable Kac-Moody algebra. This is done by replacing the action of $X$ on $V$ with an endomorphism that preserves weight spaces, but which is bar-linear instead of linear.

1. INTRODUCTION

Let $g$ be a finite type complex simple Lie algebra, and let $U_q(g)$ be the corresponding quantized universal enveloping algebra. In [KR] and [LS], Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal R-matrix

$$R = (X^{-1} \otimes X^{-1}) \Delta(X),$$

where $X$ belongs to a completion of $U_q(g)$. The element $X$ is constructed using the braid group element $T_{w_0}$ corresponding to the longest word of the Weyl group, so only makes sense when $g$ is of finite type.

The element $X$ defines a vector space endomorphism $X_V$ on each representation $V$, and in fact $X$ is defined by this system $\{X_V\}$ of endomorphisms. With this point of view, Equation (1) is equivalent to the claim that, for any finite dimensional representations $V$ and $W$ and $u \in V \otimes W$,

$$R(u) = (X_V^{-1} \otimes X_W^{-1})X_V \otimes W(u).$$

In the present work we replace $X_V$ with an endomorphism $\Theta_V$ which preserves weight spaces. We show that, for any symmetrizable Kac-Moody algebra $g$, and any integrable highest weight representations $V$ and $W$ of $U_q(g)$, the action of the universal R-matrix on $u \in V \otimes W$ is given by

$$R(u) = (\Theta_V^{-1} \otimes \Theta_W^{-1})\Theta_V \otimes W(u).$$

There is a technical difficulty because $\Theta_V$ is not linear over the base field $\mathbb{Q}(q)$, but instead is compatible with the automorphism of $\mathbb{Q}(q)$ which inverts $q$. For this reason $\Theta_V$ depends on a choice of a “bar involution” on $V$. To make Equation (3) precise we define a bar involution on $V \otimes W$ in terms of chosen involutions of $V$ and $W$, and then show that the composition $(\Theta_V^{-1} \otimes \Theta_W^{-1})\Theta_V \otimes W$ does not depend on any choices.

The system of endomorphisms $\Theta$ was previously studied in [T], where it was used to construct the universal R-matrix when $g$ is of finite type. Essentially we have extended this previous work to include all symmetrizable Kac-Moody algebras. However, the action of $\Theta$ on a tensor product is defined differently here than in [T], so the constructions of $R$ are a-priori not identical, and we have not in fact proven that the construction in [T] gives the universal R-matrix in all cases.

This note is organized as follows. In Section 2 we establish notation and review some background material. In Section 3 we construct the system of endomorphisms $\Theta$. In Section 4 prove our main
Theorem (Theorem 4.1), which simply says that our construction gives the universal $R$-matrix in all cases. In Section 5 we discuss two questions which motivated this work.

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2. Background

2.1. Conventions. We first fix some notation. For the most part we follow conventions from [CP].

- $\mathfrak{g}$ is a complex simple Lie algebra with Cartan algebra $\mathfrak{h}$ and Cartan matrix $A = (a_{ij})_{i,j \in I}$.
- $(\cdot, \cdot)$ denotes the paring between $\mathfrak{h}$ and $\mathfrak{h}^\ast$ and $(\cdot, \cdot)$ denotes the usual symmetric bilinear form on either $\mathfrak{h}$ or $\mathfrak{h}^\ast$. Fix the usual bases $a_i$ for $\mathfrak{h}^\ast$ and $H_i$ for $\mathfrak{h}$, and recall that $(H_i, a_j) = a_{ij}$.
- $d_i = (\alpha_i, \alpha_i)/2$, so that $(H_i, H_j) = d_j^{-1} a_{ij}$.
- $\rho$ is the weight satisfying $(\alpha_i, \rho) = d_i$ for all $i$.
- $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra associated to $\mathfrak{g}$, generated over $\mathbb{Q}(q)$ by $E_i$ and $F_i$ for all $i \in I$, and $K_w$ for $w$ in the co-weight lattice of $\mathfrak{g}$. As usual, let $K_i = K_{H_i}$. We use conventions as in [CP]. For convenience, we recall the exact formula for the coproduct:

$$
\begin{align*}
\Delta E_i &= E_i \otimes K_i + 1 \otimes E_i \\
\Delta F_i &= F_i \otimes 1 + K_i^{-1} \otimes F_i \\
\Delta K_i &= K_i \otimes K_i
\end{align*}
$$

- We in fact need to adjoin a fixed $k^{th}$ root of $q$ to $\mathbb{Q}(q)$, where $k$ is twice the size of the weight lattice mod the root lattice. We denote this by $q^{1/k}$.
- $V_\lambda$ is the irreducible representation of $U_q(\mathfrak{g})$ with highest weight $\lambda$.
- $v_\lambda$ is a highest weight vector of $V_\lambda$.
- A vector $v$ in a representation $V$ is called singular if $E_i(v) = 0$ for all $i \in I$.
- $V(\mu)$ denotes the $\mu$ weight space of $V$.
- Throughout, a representation of $U_q(\mathfrak{g})$ means a type 1 integrable highest weight representation.

2.2. The $R$-matrix. We briefly recall the definition of a universal $R$-matrix, and the related notion of a braiding.

**Definition 2.1.** A braided monoidal category is a monoidal category $\mathcal{C}$, along with a natural system of isomorphisms $\sigma_{V,W}^{br} : V \otimes W \to W \otimes V$ for each pair $V, W \in \mathcal{C}$, such that, for any $U, V, W \in \mathcal{C}$, the following two equalities hold:

$$
\begin{align*}
\sigma_{U,V,W}^{br} \otimes \Id &\circ \Id \otimes \sigma_{V,W}^{br} = \sigma_{U \otimes V,W}^{br} \\
\Id &\circ \sigma_{U,V,W}^{br} \otimes \sigma_{U,V \otimes W}^{br} = \sigma_{U,V,W}^{br} \otimes \Id
\end{align*}
$$

The system $\sigma^{br} := \{\sigma_{V,W}^{br}\}$ is called a braiding on $\mathcal{C}$.

Let $U_q(\mathfrak{g}) \otimes \widehat{U_q(\mathfrak{g})}$ be the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in the weak topology defined by all matrix elements of $V_\lambda \otimes V_\mu$, for all ordered pairs of dominant integral weights $(\lambda, \mu)$.

**Definition 2.2.** A universal $R$-matrix is an element $R$ of $U_q(\mathfrak{g}) \otimes \widehat{U_q(\mathfrak{g})}$ such that $\sigma_{V,W}^{br} := \text{Flip} \circ R$ is a braiding on the category of $U_q(\mathfrak{g})$ representations. Equivalently, an element $R$ is a universal $R$-matrix if it satisfies the following three conditions

(i) For all $u \in U_q(\mathfrak{g})$, $R\Delta(u) = \Delta^{op}(u)R$.

(ii) $(\Delta \otimes 1)R = R_{13}R_{23}$, where $R_{ij}$ mean $R$ placed in the $i$ and $j$th tensor factors.

(iii) $(1 \otimes \Delta)R = R_{13}R_{12}$. 


The following theorem is central to the theory of quantized universal enveloping algebra. See [CP] for a discussion when \( \mathfrak{g} \) is of finite type, and [L] for the general case.

**Proposition 2.3.** Let \( \mathfrak{g} \) be a symmetrizable Kac-Moody algebra. Then \( U_q(\mathfrak{g}) \) has a unique universal \( R \)-matrix of the form

\[
R = A \left( 1 \otimes 1 + \sum_{\text{positive integral weights } \beta} X_\beta \otimes Y_\beta \right),
\]

where \( X_\beta \) has weight \( \beta \), \( Y_\beta \) has weight \(-\beta\), and for all \( v \in V \) and \( w \in W \), \( A(v \otimes w) = q^{\langle \text{wt}(v), \text{wt}(w) \rangle} \).

2.3. **Constructing isomorphisms using systems of endomorphisms.** In this section we review a method for constructing natural systems of isomorphisms \( \sigma_{V,W} : V \otimes W \to W \otimes V \) for representations \( V \) and \( W \) of \( U_q(\mathfrak{g}) \). This idea was used by Henriques and Kamnitzer in [HK], and was further developed in [KT2]. The data needed is:

(i) An algebra automorphism \( C_\xi \) of \( U_q(\mathfrak{g}) \) which is also a coalgebra anti-automorphism.

(ii) A natural system of invertible (vector space) endomorphisms \( \xi_V \) of each representation \( V \) of \( U_q(\mathfrak{g}) \) such that the following diagram commutes for all \( V \):

\[
\begin{array}{ccc}
V & \xrightarrow{\xi_V} & V \\
\bigcirc & & \bigcirc \\
U_q(\mathfrak{g}) & \xrightarrow{c_\xi} & U_q(\mathfrak{g}).
\end{array}
\]

It follows immediately from the definition of coalgebra anti-automorphism that

\[
\sigma^\xi := \text{Flip} \circ (\xi_V^{-1} \otimes \xi_W^{-1}) \circ \xi_V \otimes W
\]

is an isomorphism of \( U_q(\mathfrak{g}) \) representations from \( V \otimes W \) to \( W \otimes V \).

In the current work we require a little more freedom: we will sometimes use automorphisms \( C_\xi \) of \( U_q(\mathfrak{g}) \) which are not linear over \( \mathbb{C}(q) \), but instead are bar-linear (i.e. invert \( q \)). This causes some technical difficulties, which we deal with in Section 3.

**Comment 2.4.** To describe the data \( (C_\xi, \xi) \), it is sufficient to describe \( C_\xi \), and the action of \( \xi_{V_\lambda} \) on any one vector \( v \) in each irreducible representation \( V_\lambda \). This is usually more convenient then describing \( \xi_{V_\lambda} \) explicitly. Of course, the choice of \( C_\xi \) imposes a restriction on the possibilities for \( \xi_{V_\lambda} \), so when we give a description of \( \xi \) in this way we are always claiming that the action on our chosen vector in each \( V_\lambda \) is compatible with \( C_\xi \).

2.4. **A useful lemma.** Let \((V_\lambda, v_\lambda)\) and \((V_\mu, v_\mu)\) be irreducible representations with chosen highest weight vectors. Every vector \( u \in V_\lambda \otimes V_\mu \) can be written as

\[
u = v_\lambda \otimes c_0 + b_{k-1} \otimes c_1 + \ldots + b_1 \otimes c_{k-1} + b_0 \otimes v_\mu,
\]

where, for \( 0 \leq j \leq k-1 \), \( b_j \) is a weight vector of \( V_\lambda \) of weight strictly less than \( \lambda \), and \( c_j \) a weight vector of \( V_\mu \) of weight strictly less than \( \mu \). Furthermore, the vectors \( b_0 \in V_\lambda \) and \( c_0 \in V_\mu \) are uniquely determined by \( u \). Thus we can define projections from \( V_\lambda \otimes V_\mu \) to \( V_\lambda \) and \( V_\mu \) as follows:

**Definition 2.5.** The projections \( p_{V_\lambda, V_\mu}^1 : V_\lambda \otimes V_\mu \to V_\lambda \) and \( p_{V_\lambda, V_\mu}^2 : V_\lambda \otimes V_\mu \to V_\mu \) are given by, for all \( u \in V_\lambda \otimes V_\mu \),

\[
p_{V_\lambda, V_\mu}^1(u) := b_0
\]

\[
p_{V_\lambda, V_\mu}^2(u) := c_0.
\]
Lemma 2.6. Let $S_{\lambda,\mu}$ be the space of singular vectors in $V_\lambda \otimes V_\mu$. The restrictions of the maps $p^{1}_{\lambda,\mu}$ and $p^{2}_{\lambda,\mu}$ from Definition 2.5 to $S_{\lambda,\mu}$ are injective.

Proof. We prove the Lemma only for $p^{2}_{\lambda,\mu}$, since the proof for $p^{1}_{\lambda,\mu}$ is completely analogous. Let $c_1, \ldots, c_m$ be a weight basis for $V_\mu$. Let $u$ be a singular vector of $V_\lambda \otimes V_\mu$ of weight $\nu$. Then $u$ can be written uniquely as

$$u = \sum_{j=1}^{m} v_j \otimes c_j,$$

where each $v_j$ is a weight vector in $V_\lambda$. Let $\gamma$ be a maximal weight such that there is some $j$ with $\text{wt}(v_j) = \gamma$ and $v_j \neq 0$. It suffices to show that $\gamma = \lambda$, so assume for a contradiction that it does not. Then $v_j$ is not a highest weight vector, so $E_i(v_j) \neq 0$ for some $i$. But then

$$E_i(u) = \sum_{\text{wt}(v_j, \nu) = \gamma} E_i(v_j) \otimes c_j + \text{terms whose first factors have weight strictly less than } \gamma + \alpha_i.$$

Since the $c_j$ are linearly independent and $E_i(v_j) \neq 0$ for some $j$ with $\text{wt}(v_j) = \gamma$, this implies that $E_i(u) \neq 0$, contradicting the fact that $u$ is a singular vector.

3. Constructing the system of endomorphisms $\Theta$

Constructing and studying $\Theta = \{\Theta_V\}$ is the technical heart of this work. As we mentioned in the introduction, $\Theta_V$ is bar linear instead of linear, which makes it more difficult to choose a normalization. To get around this, we introduce the notion of a bar involution $\text{bar}_V$ on $V$, and actually define $\Theta$ on the category of representations with a chosen bar involution. We then define a tensor product on this new category, and show that $(\Theta_{V,\text{bar}_V}^{-1} \otimes \Theta_{W,\text{bar}_W}^{-1}) \circ \Theta_{(V,\text{bar}_V),(W,\text{bar}_W)}$ does not depend on the choices of $\text{bar}_V$ and $\text{bar}_W$. The real work is in defining this tensor product, which essentially amounts to defining a bar involution on $V \otimes W$ in terms of bar involutions $\text{bar}_V$ and $\text{bar}_W$.

3.1. Bar involution. The following $\mathbb{Q}$ algebra involution of $U_q(\mathfrak{g})$ has been studied in several places, for example [K, Section 1.3], and is usually called bar involution. We use the notation $C_{\text{bar}}$ because we will also work with bar involutions $\text{bar}_V$ on representations $V$, which are compatible with $C_{\text{bar}}$ in the sense of Equation (8).

Definition 3.1. $C_{\text{bar}} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the $\mathbb{Q}$-algebra involution defined by

$$\begin{cases} C_{\text{bar}}q = q^{-1} \\
C_{\text{bar}}K_i = K_i^{-1} \\
C_{\text{bar}}E_i = E_i \\
C_{\text{bar}}F_i = F_i. \end{cases}$$

It is perhaps useful to imagine that $q$ is specialized to a complex number on the unit circle (although not a root of unity), so that $C_{\text{bar}}$ is conjugate linear.

Definition 3.2. Let $V$ be a representation of $U_q(\mathfrak{g})$. A bar involution on $V$ is a $\mathbb{Q}$-linear involution $\text{bar}_V$ such that
(i) \( \text{bar}_V \) is compatible with \( C_{\text{bar}} \) in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\text{bar}_V} & V \\
\bigcirc & & \bigcirc \\
\end{array}
\]

\[
U_q(\mathfrak{g}) \xrightarrow{C_{\text{bar}}} U_q(\mathfrak{g}).
\]

(ii) Let \( V^{inv} = \{ v \in V \text{ such that } \text{bar}_V(v) = v \} \). Then \( V = \mathbb{Q}(q) \otimes Q^{inv} \).

**Comment 3.3.** It is straightforward to check that \( C_{\text{bar}}^2 \) is the identity. Along with condition (ii), this implies that \( \text{bar}_V^2 \) is the identity, so the term “involution” is justified.

**Comment 3.4.** When it does not cause confusion we will denote \( \text{bar}_V(v) \) by \( \bar{v} \).

**Proposition 3.5.** Fix \( \lambda \) and a highest weight vector \( v_\lambda \in V_\lambda \). There is a unique bar involution \( \text{bar}_{(V_\lambda,v_\lambda)} \) on \( V_\lambda \) such that \( \text{bar}_{(V_\lambda,v_\lambda)}(v_\lambda) = v_\lambda \).

**Proof.** Recall that \( V_\lambda \) has a basis consisting of various \( F_{i_k} \cdots F_{i_1} v_\lambda \). All of these vectors must be fixed by any bar involution preserving \( v_\lambda \), so there is at most one possibility. On the other hand, it is clear that the unique \( \mathbb{Q}(q) \)-linear map sending \( f(q) F_{i_k} \cdots F_{i_1} v_\lambda \) to \( f(q^{-1}) F_{i_k} \cdots F_{i_1} v_\lambda \) for each of these basis vectors is a bar involution. \( \square \)

**Corollary 3.6.** Every representation \( V \) has a (non-unique) bar involution \( \text{bar}_V \).

**Proof.** Choose a decomposition of \( V \) into irreducible components, and a highest weight vector in each irreducible component, then use Proposition 3.5. \( \square \)

**Definition 3.7.** Fix \( (V, \text{bar}_V) \) and \( (W, \text{bar}_W) \), where \( \text{bar}_V \) and \( \text{bar}_W \) are involutions of \( V \) and \( W \) compatible with \( C_{\text{bar}} \). Let \( (\text{bar}_V \otimes \text{bar}_W) \) be the vector space involution on \( V \otimes W \) defined by \( f(q)v \otimes w \rightarrow f(q^{-1}) \bar{v} \otimes \bar{w} \) for all \( f(q) \in \mathbb{Q}(q) \) and \( v \in V, w \in W \).

**Comment 3.8.** It is straightforward to check that the action of \( (\text{bar}_V \otimes \text{bar}_W) \) on a vector in \( V \otimes W \) does not depend on its expression as a sum of elements of the form \( f(q)v \otimes w \). The resulting map is a \( \mathbb{Q}(q) \)-linear involution.

**Definition 3.9.** Fix \( u \in V_\lambda \otimes V_\mu \) a weight vector of weight \( \nu \). Define \( v^\beta \) for each weight \( \beta \) as the unique element of \( V_\lambda(\nu - \beta) \otimes V_\mu(\beta) \) such that

\[
u = \sum_{\beta} v^\beta.
\]

**Lemma 3.10.** Fix \( (V_\lambda, \text{bar}_{V_\lambda}) \) and \( (V_\mu, \text{bar}_{V_\mu}) \). Let \( v_\nu \) be a singular weight vector in \( V_\lambda \otimes V_\mu \), and write

\[
v_\nu = \sum_{j=1}^N b_j \otimes c_j,
\]

where each \( b_j \) is a weight vector of \( V_\lambda \), and each \( c_j \) is a weight vector of \( V_\mu \). Then

\[
\text{bar}(v_\nu) := \sum_{j=0}^N q^{(\mu,\nu)-(\text{wt}(c_j),\text{wt}(c_j))}+2(\mu-\text{wt}(c_j),\rho)b_j \otimes c_j
\]

is also singular.
Proof. Fix $i \in I$. The vector $v_{\beta}$ is singular, so $E_i v_{\beta} = 0$ and hence $(E_i v_{\beta})^\beta = 0$ for all $\beta$. Then:

$$(19) \quad 0 = (E_i v_{\beta})^\beta = \sum_{\text{wt}(c_j) = \beta} q^{(\beta,\alpha_i)} E_i b_j \otimes c_j + \sum_{\text{wt}(c_j) = \beta - \alpha_i} b_j \otimes E_i c_j.$$ 

Using Equation (18):

$$(20) \quad (E_i \text{bar}(v_{\beta}))^\beta = \sum_{\text{wt}(c_j) = \beta} q^{(\mu,\nu)} (\beta,\beta) + 2(\mu - \alpha_i,\rho) q^{(\beta,\alpha_i)} E_i b_j \otimes c_j$$
\begin{equation}
+ \sum_{\text{wt}(c_j) = \beta - \alpha_i} q^{(\mu,\nu)} (\beta - \alpha_i,\beta - \alpha_i) + 2(\mu - \alpha_i,\rho) b_j \otimes E_i c_j
\end{equation}

$$(21) \quad = q^{(\mu,\nu)} (\beta - \alpha_i,\beta - \alpha_i) + 2(\mu - \alpha_i,\rho) \times$$

$$\sum_{\text{wt}(c_j) = \beta} q^{(-\beta,\alpha_i)} E_i b_j \otimes c_j + \sum_{\text{wt}(c_j) = \beta - \alpha_i} b_j \otimes E_i c_j,$$

$$(22) \quad = q^{(\mu,\nu)} (\beta - \alpha_i,\beta - \alpha_i) + 2(\mu - \alpha_i,\rho) \text{bar}(\text{bar}_\lambda \otimes \text{bar}_\rho)(E_i v_{\beta})^\beta,$$

where $(\text{bar}_\lambda \otimes \text{bar}_\rho)$ is the involution from Definition 3.7. But $E_i (v_{\beta})^\beta = 0$, so we see that $E_i (v_{\beta})^\beta = 0$. Since this holds for all $i$ and all $\beta$, $\text{bar}(v_{\beta})$ is singular. $\square$

**Definition 3.11.** Let $\text{bar}(V, v_{\lambda}) \otimes (V, v_{\mu})$ be the unique involution on $V \otimes V$ which agrees with the involution bar from Lemma 3.10 on singular vectors, and is compatible with $C_{\text{bar}}$.

**Lemma 3.12.** $\text{bar}(V, v_{\lambda}) \otimes (V, v_{\mu})$ is a bar involution.

**Proof.** Definition 3.2 part (i) follows immediately from the definition of $\text{bar}(V, v_{\lambda}) \otimes (V, v_{\mu})$. To establish Definition 3.2 part (ii), it suffices to show that there is a basis for the space $S_{\lambda,\mu}$ of singular vectors of $V \otimes V$ which is fixed by $\bar{v}_{(V, v_{\lambda}) \otimes (V, v_{\mu})}$. Since $V = \mathbb{Q}(q) \otimes V_{\text{inv}}$, there is a basis for $S_{\lambda,\mu}$ consisting of elements of $V_{\text{inv}} \otimes \mathbb{Q} V$. Using Lemma 2.6, we see that there is a basis for $S_{\lambda,\mu}$ consisting of vectors of the form

$$(23) \quad v_{\lambda} \otimes c_0 + \cdots + b_0 \otimes v_{\mu},$$

where $b_0 = b_0$ and the missing terms are all of the form $b \otimes c$ with $\text{wt}(c) < \mu$. By Definition 3.11 and Lemma 2.6, this vector is invariant under $\text{bar}(V, v_{\lambda}) \otimes (V, v_{\mu})$. $\square$

In light of Definition 3.2 part (ii), we can extend Definition 3.11 by naturality to construct a bar-involution on $(V, \text{bar}_V) \otimes (W, \text{bar}_W)$ in terms of any bar-involutions of $V$ and $W$.

### 3.2. The system of endomorphisms $\Theta$

Consider the $\mathbb{Q}$-algebra automorphism $C_\Theta$ of $U_q(g)$:

$$C_\Theta(E_i) = E_i K_i^{-1},$$

$$C_\Theta(F_i) = K_i F_i,$$

$$C_\Theta(K_i) = K_i^{-1},$$

$$C_\Theta(q) = q^{-1}.$$

Notice that $C_\Theta$ is not linear over $\mathbb{Q}(q)$, but instead inverts $q$. One can easily check that $C_\Theta$ is a $\mathbb{Q}$ algebra involution, and that it is also a coalgebra anti-involution.

**Definition 3.13.** Fix a representation $V$ with a bar involution $\text{bar}_V$. Then $\Theta_{V, \text{bar}_V}$ is the $\mathbb{Q}$ linear endomorphism of $V$ defined by

$$(25) \quad \Theta_{V, \text{bar}_V}(v) = q^{-(\text{wt}(v),\text{wt}(v))/2+(\text{wt}(v),\rho) \text{bar}_V(v)}.$$
Comment 3.14. Using Definitions 3.1, one can see that, for any irreducible $V_{\lambda} \subset V$, $\Theta_{V,\bar{V}_{\lambda}}$ restricts to an endomorphism of $V_{\lambda}$.

Comment 3.15. There are sometimes weights $\lambda$ for which $-(\lambda, \lambda)/2 + (\lambda, \rho)$ is not an integer. However, it is always a multiple of $1/k$ where $k$ is twice the size of the weight lattice mod the root lattice. It is for this reason that we adjoin $q^{1/k}$ to the base field.

Lemma 3.16. The following diagram commutes

$$
\begin{array}{ccc}
V & \xrightarrow{\Theta_{\lambda}} & V \\
\cup & & \cup \\
U_q(\mathfrak{g}) & \xrightarrow{c_{\Theta}} & U_q(\mathfrak{g}).
\end{array}
$$

Proof. It is sufficient to check that $C_{\Theta}(X)\Theta_{V}(v) = \Theta_{V}(Xv)$, where $X = E_i$ or $F_i$. We do the case of $F_i$ and leave $E_i$ as an exercise. Fix $v \in V$.

1. $\Theta_{V}(F_i v) = q^{-(\text{wt}(F_i v), \text{wt}(F_i v))/2 + (\text{wt}(F_i v), \rho)} \bar{\Theta}_{V}(F_i v)\bar{\lambda}
\begin{align*}
\Theta_{V}(F_i v) &= q^{-(\text{wt}(v) - \alpha, \text{wt}(v) - \alpha)/2 + (\text{wt}(v) - \alpha, \rho)} F_i \bar{\Theta}_{V}(v) \\
&= q^{(\alpha, \text{wt}(v) - \alpha)} q^{-(\text{wt}(v), \text{wt}(v))/2 + (\text{wt}(v), \rho)} F_i \bar{\Theta}_{V}(v) \\
&= K_i F_i q^{-(\text{wt}(v), \text{wt}(v))/2 + (\text{wt}(v), \rho)} \bar{\Theta}_{V}(v) \\
&= C_{\Theta}(F_i) \Theta_{V}(v).
\end{align*}

where for Equation (29) we have used the fact that $(\alpha_i, \alpha_i)/2 = (\alpha_i, \rho) = d_i$. \hfill \Box

Definition 3.17. Fix two representations with bar involutions $(V, \bar{V}_{\lambda})$ and $(W, \bar{W}_{\lambda})$. We set $\Theta_{V,\bar{V}_{\lambda}} \otimes (W, \bar{W}_{\lambda})$ to be the $\mathbb{Q}$ linear endomorphism of $V \otimes W$ defined by, for all $u \in V \otimes W$,

$$
\Theta_{V,\bar{V}_{\lambda}} \otimes (W, \bar{W}_{\lambda})(u) = q^{-\text{wt}(u), \text{wt}(u))/2 + (\text{wt}(u), \rho)} \bar{\Theta}_{V,\bar{V}_{\lambda}} \otimes (W, \bar{W}_{\lambda}).
$$

Comment 3.18. By Lemma 3.12, $\bar{V}_{\lambda} \otimes (W, \bar{W}_{\lambda})$ is a bar involution on $V \otimes W$, so by Lemma 3.16, $\Theta_{V,\bar{V}_{\lambda}} \otimes (W, \bar{W}_{\lambda})$ is compatible with $C_{\Theta}$.

4. Main Theorem

Theorem 4.1. $(\Theta_{V,\bar{V}_{\lambda}} \otimes \Theta_{W,\bar{W}_{\lambda}}) \otimes (V, \bar{V}_{\lambda}) \otimes (W, \bar{W}_{\lambda})$ acts on $V \otimes W$ as the standard $R$-matrix. This holds independent of the choice of bar involutions $\bar{V}_{\lambda}$ and $\bar{W}_{\lambda}$.

Proof. We will actually prove the equivalent statement that

$$
\sigma^{\Theta} := \text{Flip} \circ (\Theta_{V,\bar{V}_{\lambda}} \otimes \Theta_{W,\bar{W}_{\lambda}}) \otimes (V, \bar{V}_{\lambda}) \otimes (W, \bar{W}_{\lambda})
$$

acts on $V \otimes W$ as the standard braiding $\text{Flip} \circ R$. By Lemma 3.16 and the fact that $C_{\Theta}$ is a $\mathbb{Q}$ coalgebra anti-automorphism, the following diagram commutes:

$$
\begin{array}{ccc}
V \otimes W & \xrightarrow{\Theta_{V,\bar{V}_{\lambda}} \otimes (W, \bar{W}_{\lambda})} & V \otimes W \\
\cup & & \cup \\
U_q(\mathfrak{g}) & \xrightarrow{c_{\Theta}} & U_q(\mathfrak{g}).
\end{array}
$$

In particular, $\sigma^{\Theta} : V \otimes W \rightarrow W \otimes V$ is an isomorphism. Thus it suffices to show that $\sigma^{\Theta}(v_{\mu}) = \text{Flip} \circ R(v_{\mu})$ for every singular weight vector $v_{\mu} \in V \otimes W$. By naturality it is enough to consider the case when $V$ and $W$ are irreducible, so let $v_{\mu}$ be a singular vector in $V_{\lambda} \otimes V_{\mu}$. Write

$$
v_{\mu} = b_{\lambda} \otimes c_{0} + b_{k-1} \otimes c_{1} + \ldots + b_{1} \otimes c_{k-1} + b_{0} \otimes b_{\mu},
$$
where for $0 \leq j \leq k - 1$, $b_j$ is a weight vector of $V_\mu$ of weight strictly less then $\mu$. By Definitions 3.11 and 3.13,

$$\sigma^\Theta(v_\nu) = \text{Flip} \circ (\Theta_{V_\lambda, \text{bar}_V}^{-1} \otimes \Theta_{V_\nu, \text{bar}_V}^{-1}) \Theta_{(V_\lambda, \text{bar}_V)}(\nu, \text{bar}_V) (\cdots + b_0 \otimes b_\mu)$$

(36) $$= \text{Flip} \circ (\Theta_{V_\lambda, \text{bar}_V}^{-1} \otimes \Theta_{V_\nu, \text{bar}_V}^{-1}) (q^{-1}(\mu + \nu, \nu + \mu) / 2 \pm (\mu + \nu, \nu)) (\cdots + b_0 \otimes b_\mu)$$

(37) $$= q^{-1}(\nu, \nu - \nu) / 2 - (\nu, \nu) / 2 + (\nu, \nu - \nu) / 2 b_\mu \otimes b_0 + \cdots$$

(38) $$= q^{\nu, \nu - \nu} b_\mu \otimes b_0 + \cdots$$

(39) $$\text{where } \cdots \text{ always represents terms where the factor coming from } V_\mu \text{ has weight strictly less then } \mu.$$ 

It follows immediately from Proposition 2.3 that

$$\text{Flip} \circ R(v_\nu) = q^{\nu, \nu - \nu} b_\mu \otimes b_0 + \cdots$$

where again $\cdots$ represents terms of the form $c \otimes b$ where $\nu, \nu - \nu < \mu$. Both $\sigma^\Theta(v_\nu)$ and $\text{Flip} \circ R(v_\nu)$ are singular vectors in $V_\mu \otimes V_\lambda$, so by Lemma 2.6 they are equal. \qed

**Comment 4.2.** The above proof works independent of the choice of $\text{bar}_V$ and $\text{bar}_W$. One can also see directly that $\sigma^\Theta$ does not depend on these choices. Restrict to the irreducible case, and notice that by Lemma 3.5, $\sigma^\Theta$ depends only the a choice of highest weight vectors $v_\lambda$ and $v_\mu$. It is straightforward to check that rescaling these vectors has no effect on $\sigma^\Theta$.

**Comment 4.3.** One can check that $\Theta_V$ is an involution of $\mathbb{Q}$ vector spaces, so the inverses in the statement of Theorem 4 are in some sense unnecessary. We include them because $\Theta_V$ should really be thought of as an isomorphism between $V$ and the module which is $V$ as a $\mathbb{Q}$ vector space, but with the action of $U_q(g)$ twisted by $C_{\Theta}$. We have not specified the action of $\Theta$ on this new module. The way the formula is written, $\Theta$ is always acting on $V, W$ or $V \otimes W$ with the usual action, where it has been defined.

## 5. Future directions

We have two main motivations for developing our formula for the R-matrix.

**Motivation 1.** In work with Joel Kamnitzer [KT2], we showed that Drinfeld’s unitarized R-matrix $\tilde{R}$ (see [D]) respects crystal basis (up to some signs). Composing with Flip, we see that $\tilde{R}$ descends to a crystal map from $B \otimes C$ to $C \otimes B$, which is fact agrees with the crystal commutator defined in [HK]. We make extensive use of Equation (1), so our methods are only valid in the finite type case. However Drinfeld’s unitarized R-matrix is defined in the symmetrizable Kac-Moody case, as is the crystal commutator (see [KT1] and [S]). We hope that the formula given in Theorem 4.1 will help us to extend some of the results in [KT2] to the symmetrizable Kac-Moody case.

**Motivation 2.** Recall that the action of the braiding Flip $\circ R$ on $V \otimes W$ can be drawn diagrammatically as passing a string labeled $V$ over a string labeled $W$. If we use flat ribbons in place of strings, as it is often convenient to do, one can consider the following isotopy:
Roughly, if one interprets twisting a ribbon by 180 degrees as $X$, and twisting two ribbon together as at the bottom on the right side as $\text{Flip} \circ \Delta(X)$, the two sides of this isotopy correspond to the two sides of Equation (1), written as

\begin{equation}
\text{Flip} \circ R = \text{Flip} \circ (X^{-1} \otimes X^{-1}) \Delta(X) = (X^{-1} \otimes X^{-1}) \circ \text{Flip} \circ \Delta(X).
\end{equation}

In work with Noah Snyder [ST], we make this precise. One should be able to use our new formula to give a precise interpretation of “twisting a ribbon by 180 degrees” in the symmetrizable Kac-Moody case. It is for this reason that we use the term “half twist type formula” in our title.

**References**


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